

Brauer algebras of type C

Arjeh M. Cohen, Shoumin Liu, Shona Yu

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Abstract

For each $n \geq 2$, we define an algebra satisfying many properties that one might expect to hold for a Brauer algebra of type C_n . The monomials of this algebra correspond to scalar multiples of symmetric Brauer diagrams on $2n$ strands. The algebra is shown to be free of rank the number of such diagrams and cellular, in the sense of Graham and Lehrer.

KEYWORDS: associative algebra, Birman–Murakami–Wenzl algebra, BMW algebra, Brauer algebra, cellular algebra, Coxeter group, Temperley–Lieb algebra, root system, semisimple algebra, word problem in semigroups

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CORRESPONDING AUTHOR: Arjeh M. Cohen, Department of Mathematics and Computer Science, Eindhoven University of Technology, POBox 513, 5600 MB Eindhoven, The Netherlands, email: A.M.Cohen@tue.nl

1 Introduction

It is well known that the Coxeter group of type C_n arises from the Coxeter group of type A_{2n-1} as the subgroup of all elements fixed by a Coxeter diagram automorphism. Crisp [6] showed that the Artin group of type C_n arises in a similar fashion from the Artin group of type A_{2n-1} . In this paper, we study the subalgebra of the Brauer algebra $\text{Br}(A_{2n-1})$ of type A_{2n-1} (that is, the classical Brauer algebra on $2n$ strands) spanned by Brauer diagrams that are fixed by the symmetry corresponding to this diagram automorphism (see Definition 2.2). Such diagrams will be called symmetric. First, in Definition 2.1, we define the Brauer algebra of type C_n , notation $\text{Br}(C_n)$, in terms of generators and relations depending solely on the Dynkin diagram below.

$$C_n = \begin{array}{c} \circ \text{---} \circ \cdots \circ \text{---} \circ \text{---} \circ \\ n-1 \quad n-2 \quad \quad \quad 2 \quad 1 \quad 0 \end{array}$$

The distinguished generators of $\text{Br}(C_n)$ are the involutions r_0, \dots, r_{n-1} and the quasi-idempotents e_0, \dots, e_{n-1} (here, a quasi-idempotent is an element that is an idempotent up to a scalar multiple). Each defining relation concerns at most two indices, say i and j , and is determined by the diagram induced by C_n on $\{i, j\}$. The group algebra of the Coxeter group of type C_n is obtained by taking the quotient of the Brauer algebra of type C_n by the ideal generated by all quasi-idempotents e_i . It is isomorphic to the subalgebra generated by all r_i . The subalgebra generated by all e_i ($i = 0, \dots, n-1$) is isomorphic to the Temperley–Lieb algebra of type B_n defined by tom Dieck in [7].

The main result states that the algebra $\text{Br}(C_n)$ is isomorphic to the subalgebra $\text{SBr}(A_{2n-1})$ of the Brauer algebra $\text{Br}(A_{2n-1})$ linearly spanned by symmetric diagrams. In order to distinguish them from those of $\text{Br}(C_n)$, the canonical generators of the Brauer algebra of type A_{2n-1} are denoted by $R_1, \dots, R_{2n-1}, E_1, \dots, E_{2n-1}$ instead of the usual lower case letters (see Definition 2.2). Although our formal set-up is slightly more general, the algebras considered are mostly defined over the integral group ring $\mathbb{Z}[\delta^{\pm 1}]$.

Theorem 1.1. *There exists a $\mathbb{Z}[\delta^{\pm 1}]$ -algebra isomorphism*

$$\phi : \text{Br}(C_n) \longrightarrow \text{SBr}(A_{2n-1})$$

determined by $\phi(r_0) = R_n$, $\phi(r_i) = R_{n-i}R_{n+i}$, $\phi(e_0) = E_n$, and $\phi(e_i) = E_{n-i}E_{n+i}$, for $0 < i < n$. In particular, the algebra $\text{Br}(C_n)$ is free over $\mathbb{Z}[\delta^{\pm 1}]$ of rank a_{2n} , where a_n is defined by $a_0 = a_1 = 1$ and, for $n > 1$, the recursion

$$a_n = a_{n-1} + 2(n-1)a_{n-2}.$$

A closed formula for the rank of $\text{Br}(C_n)$ is

$$a_{2n} = \sum_{i=0}^n \left(\sum_{p+2q=i} \frac{n!}{p!q!(n-i)!} \right)^2 2^{n-i} (n-i)!. \quad (1.1)$$

A table of a_n for some small n is provided below.

n	0	1	2	3	4	5	6	7	8
a_n	1	1	3	7	25	81	331	1303	5937

The paper is structured as follows. In Section 2, we review the definition of a Brauer monoid of any simply laced Coxeter type and introduce the notion of a Brauer algebra of type C_n , denoted $\text{Br}(C_n)$. Section 3 reviews facts about the classical Brauer algebra, denoted $\text{Br}(A_n)$, and about admissible root sets

as presented in [4]. These sets lead towards a normal form of monomials closely related to cellularity. In Section 4, we derive elementary properties of $\text{Br}(C_n)$. Next, in Section 5, we prove that the image of ϕ is precisely the symmetric diagram subalgebra $\text{SBr}(A_{2n-1})$ of $\text{Br}(A_{2n-1})$. In Section 6 we study symmetric diagrams, the related algebra $\text{SBr}(A_{2n-1})$, and the action of the monoid of all monomials of $\text{Br}(C_n)$ on certain orthogonal root sets and a normal form for monomials in $\text{Br}(C_n)$. With these results, we are able to prove Theorem 1.1 in Section 6. Finally, in Section 7, we establish cellularity (in the sense of [10]) of the newly introduced Brauer algebras and derive some further properties.

We finish this introduction by illustrating our results with the first interesting case: $n = 2$. Consider the classical Brauer algebra $\text{Br}(A_3)$. The corresponding Brauer diagrams consist of four nodes at the top and four at the bottom together with a complete matching between these eight nodes. See Figure 1 for interpretations of R_i and E_i ($i = 1, 2, 3$). A Brauer diagram is called *symmetric* if the complete matching is not altered by the reflection of the plane whose mirror is the vertical central axis of the diagram. Clearly, $e_1 := E_1 E_3$, $e_0 := E_2$, $r_1 := R_1 R_3$, and $r_0 := R_2$ represent symmetric diagrams. Our main theorem implies that the subalgebra of $\text{Br}(A_3)$ generated by these diagrams has a presentation on these four generators by the relations given in Definition 2.1 for $n = 2$, and moreover it coincides with $\text{SBr}(A_3)$, the linear span of all symmetric diagrams. In fact, it is free and spanned by the following 25 monomials.

$$\begin{aligned} &1, r_0, r_1, r_0 r_1, r_1 r_0, r_1 r_0 r_1, r_0 r_1 r_0 r_1, r_0 r_1 r_0, \\ &\quad \{1, r_1\} e_0 \{1, r_1 r_0 r_1\} \{1, r_1\}, \\ &\quad \{1, r_0, e_0\} e_1 \{1, r_0, e_0\}. \end{aligned}$$

The first eight, given on the top line, span a subalgebra isomorphic to the group algebra of the Weyl group of type C_2 . This is in accordance with the construction of $\text{SBr}(A_3)$ and the fact that the Weyl group of type C_2 occurs in the Weyl group of type A_3 as the subgroup of elements fixed by a Coxeter diagram automorphism. The two-sided ideal of $\text{SBr}(A_3)$ generated by e_1 is spanned by the 9 monomials on the bottom line. Also, the complement in the ideal generated by e_0 and e_1 of the ideal generated by e_1 is spanned by the 8 monomials on the middle line. This division of the 25 spanning monomials into three parts along the above lines is strongly related to the cellular structure of $\text{SBr}(A_3)$. The subalgebra of $\text{SBr}(A_3)$ generated by e_0 and e_1 has dimension 6 and is isomorphic to the Temperley–Lieb algebra of type B_2 introduced by tom Dieck [7]. For these (and other) reasons, we name $\text{SBr}(A_3)$ *the Brauer algebra of type C_2* . Remarkably, the Temperley–Lieb algebra of type B_2 defined by Graham [9] is 7-dimensional and tom Dieck’s

version is a quotient algebra thereof, but we have not found a natural extension of Graham's algebra to an object deserving the name *Brauer algebra of type C_2* .

2 Definitions

In this section, we give precise definitions of the algebras and the homomorphism ϕ appearing in Theorem 1.1. All rings and algebras given are unital and associative.

Definition 2.1. Let R be a commutative ring with invertible element δ . For $n \in \mathbb{N}$, the *Brauer algebra of type C_n over R with loop parameter δ* , denoted by $\text{Br}(C_n, R, \delta)$, is the R -algebra generated by r_0, r_1, \dots, r_{n-1} and e_0, e_1, \dots, e_{n-1} subject to the following relations.

$$r_i^2 = 1 \quad \text{for any } i \quad (2.1)$$

$$r_i e_i = e_i r_i = e_i \quad \text{for any } i \quad (2.2)$$

$$e_i^2 = \delta^2 e_i \quad \text{for } i > 0 \quad (2.3)$$

$$e_0^2 = \delta e_0 \quad (2.4)$$

$$r_i r_j = r_j r_i, \quad \text{for } i \not\sim j \quad (2.5)$$

$$e_i r_j = r_j e_i, \quad \text{for } i \not\sim j \quad (2.6)$$

$$e_i e_j = e_j e_i, \quad \text{for } i \not\sim j \quad (2.7)$$

$$r_i r_j r_i = r_j r_i r_j, \quad \text{for } i \sim j \text{ with } i, j > 0 \quad (2.8)$$

$$r_j r_i e_j = e_i e_j, \quad \text{for } i \sim j \text{ with } i, j > 0 \quad (2.9)$$

$$r_i e_j r_i = r_j e_i r_j, \quad \text{for } i \sim j \text{ with } i, j > 0 \quad (2.10)$$

$$r_1 r_0 r_1 r_0 = r_0 r_1 r_0 r_1 \quad (2.11)$$

$$r_1 r_0 e_1 = r_0 e_1 \quad (2.12)$$

$$r_1 e_0 r_1 e_0 = e_0 e_1 e_0 \quad (2.13)$$

$$(r_1 r_0 r_1) e_0 = e_0 (r_1 r_0 r_1) \quad (2.14)$$

$$e_1 r_0 e_1 = \delta e_1 \quad (2.15)$$

$$e_1 e_0 e_1 = \delta e_1 \quad (2.16)$$

$$e_1 r_0 r_1 = e_1 r_0 \quad (2.17)$$

$$e_1 e_0 r_1 = e_1 e_0 \quad (2.18)$$

Here $i \sim j$ means that i and j are adjacent in the Dynkin diagram C_n . If $R = \mathbb{Z}[\delta^{\pm 1}]$ we write $\text{Br}(C_n)$ instead of $\text{Br}(C_n, R, \delta)$ and speak of *the Brauer algebra of type C_n* . The submonoid of the multiplicative monoid of $\text{Br}(C_n)$ generated by $\delta, \delta^{-1}, \{r_i\}_{i=0}^{n-1}$, and $\{e_i\}_{i=0}^{n-1}$ is denoted by $\text{BrM}(C_n)$. It is the monoid of monomials in $\text{Br}(C_n)$ and will be called *the Brauer monoid of type C_n* .

Observe that, for a distinguished invertible element δ , the ring R can be viewed as a $\mathbb{Z}[\delta^{\pm 1}]$ -algebra and that $\text{Br}(C_n, R, \delta) \cong \text{Br}(C_n) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} R$. As a direct consequence of the above definition, the submonoid of $\text{BrM}(C_n)$ generated by $\{r_i \mid i = 0, \dots, n-1\}$ is isomorphic to the Weyl group $W(C_n)$ of type C_n .

Let us recall from [4] the definition of a Brauer algebra of simply laced Coxeter type Q . In order to avoid confusion with the above generators, the symbols of [4] have been capitalized.

Definition 2.2. Let R be a commutative ring with invertible element δ and Q be a simply laced Coxeter graph. The *Brauer algebra of type Q over R with loop parameter δ* , denoted $\text{Br}(Q, R, \delta)$, is the R -algebra generated by R_i and E_i , for each node i of Q subject to the following relations, where \sim denotes adjacency between nodes of Q .

$$R_i^2 = 1 \quad (2.19)$$

$$E_i^2 = \delta E_i \quad (2.20)$$

$$R_i E_i = E_i R_i = E_i \quad (2.21)$$

$$R_i R_j = R_j R_i, \text{ for } i \asymp j \quad (2.22)$$

$$E_i R_j = R_j E_i, \text{ for } i \asymp j \quad (2.23)$$

$$E_i E_j = E_j E_i, \text{ for } i \asymp j \quad (2.24)$$

$$R_i R_j R_i = R_j R_i R_j, \text{ for } i \sim j \quad (2.25)$$

$$R_j R_i E_j = E_i E_j, \text{ for } i \sim j \quad (2.26)$$

$$R_i E_j R_i = R_j E_i R_j, \text{ for } i \sim j \quad (2.27)$$

As before, we call $\text{Br}(Q) := \text{Br}(Q, \mathbb{Z}[\delta^{\pm 1}], \delta)$ the *Brauer algebra of type Q* and denote by $\text{BrM}(Q)$ the submonoid of the multiplicative monoid of $\text{Br}(Q)$ generated by $\delta^{-\pm 1}$ and all R_i and E_i .

For any Q , the algebra $\text{Br}(Q)$ is free over $\mathbb{Z}[\delta^{\pm 1}]$. Also, the classical Brauer algebra on $m+1$ strands is obtained when $Q = A_m$.

Remark 2.3. As a consequence of the above relations, it is straightforward to show that the following relations hold in $\text{Br}(Q)$ for all nodes i, j, k with $i \sim j \sim k$ and $i \not\sim k$ (see [4, Lemma 3.1]).

$$E_i R_j R_j = E_i E_j \quad (2.28)$$

$$R_j E_i E_j = R_i E_j \quad (2.29)$$

$$E_i R_j E_i = E_i \quad (2.30)$$

$$E_j E_i R_j = E_j R_i \quad (2.31)$$

$$E_i E_j E_i = E_i \quad (2.32)$$

$$E_j E_i R_k E_j = E_j R_i E_k E_j \quad (2.33)$$

$$E_j R_i R_k E_j = E_j E_i E_k E_j \quad (2.34)$$

Remark 2.4. In [2], Brauer gives a diagrammatic description for a basis of the Brauer algebra of type A_m . Each basis element is a diagram with $2m + 2$ dots and $m + 1$ strands, where each dot is connected by a unique strand to another dot. Here we suppose the $2m + 2$ dots have coordinates $(i, 0)$ and $(i, 1)$ in \mathbb{R}^2 with $1 \leq i \leq m + 1$. The multiplication of two diagrams is given by concatenation, where any closed loops formed are replaced by a factor of δ . The generators R_i and E_i of $\text{Br}(A_m)$ correspond to the diagrams indicated in Figure 1.

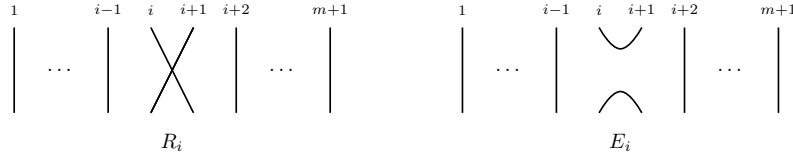


Figure 1: Brauer diagrams corresponding to R_i and E_i .

Each Brauer diagram can be written as a product of elements from $\{R_i, E_i\}_{i=1}^m$. This statement is illustrated in Figure 2.

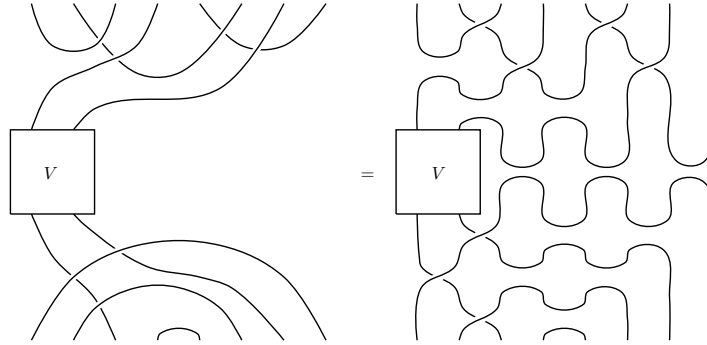


Figure 2: A Brauer diagram and a visualization of it as the product $R_2 R_5 E_1 R_3 R_6 E_2 E_4 V E_3 E_5 E_7 R_2 E_4 E_6 R_1 E_3 E_5 R_2 E_4$, where V can be either the identity or the simple crossing R_1 .

Henceforth, we identify $\text{BrM}(A_m)$ with its diagrammatic version. It makes clear that $\text{Br}(A_m)$ is a free algebra over $\mathbb{Z}[\delta^{\pm 1}]$ of rank $(m + 1)!!$,

the product of the first $m+1$ odd integers. The monomials of $\text{BrM}(A_m)$ that correspond to diagrams will be referred to as diagrams.

The map σ on the graph (or Coxeter type) A_m given by $\sigma(i) = m+1-i$ is the single nontrivial automorphism of this graph. As the presentation of $\text{Br}(Q)$ merely depends on the graph Q , the map σ induces an automorphism of $\text{Br}(A_m)$, which will also be denoted by σ . This involutory automorphism is determined by its behaviour on the generators:

$$\sigma(R_i) = R_{m+1-i}, \quad \sigma(E_i) = E_{m+1-i}.$$

The automorphism σ may be viewed simply as a reflection of the corresponding diagram about its central vertical axis.

Definition 2.5. Suppose D_1 and D_2 are diagrams in $\text{Br}(A_m)$. The diagram D_1 is *symmetric* to the diagram D_2 if D_2 is the diagram obtained by taking the reflection of D_1 about its central vertical axis. If $D_1 = D_2$, then we say D_1 is a symmetric diagram.

Hence a diagram (that is, a monomial in R_i and E_i) of $\text{Br}(A_m)$ is σ -invariant if and only if it is symmetric about its central vertical axis. A monomial in $\text{BrM}(A_m)$ is fixed by σ if and only if it represents a symmetric diagram.

Lemma 2.6. *Let $m = 2n - 1$, for some $n \in \mathbb{N}$. The number of symmetric diagrams (with respect to σ) in $\text{BrM}(A_m)$ is equal to a_{2n} , where a_n satisfies $a_0 = a_1 = 1$ and the recursion*

$$a_n = a_{n-1} + 2(n-1)a_{n-2}.$$

Proof. Fix two sets X and Y , say, of size n and a permutation τ of $X \cup Y$ of order 2 interchanging X and Y . We define a_n as the number of perfect matchings on $X \cup Y$ that are τ -invariant (that is, if $\{a, b\} \subseteq X \cup Y$ belongs to the matching, so does $\{\tau(a), \tau(b)\}$). Identifying X with the set of dots left of the vertical axis of symmetry, Y with the set of dots to the right, and τ with the permutation induced by σ , we see that that a_{2n} is the number of symmetric diagrams in $\text{BrM}(A_m)$.

It is obvious that $a_0 = a_1 = 1$. Fix $a \in X$. The number of perfect τ -invariant matchings containing $\{a, \tau(a)\}$ is equal to a_{n-1} .

Suppose that we have a perfect τ -invariant matching of $X \cup Y$ containing $\{a, b\}$ with $b \neq \tau(a)$. Then $\{\tau(a), \tau(b)\}$ is a second pair belonging to the matching. The matching induces a_{n-2} number of perfect τ -invariant matchings on $(X \cup Y) \setminus \{a, b, \tau(a), \tau(b)\}$. As there are $2n - 2$ choices of b , we find $a_n = a_{n-1} + (2n - 2)a_{n-2}$. \square

Corollary 2.7. *The linear span $\text{SBr}(A_{2n-1})$ of symmetric diagrams is a $\mathbb{Z}[\delta^{\pm 1}]$ -subalgebra of $\text{Br}(A_{2n-1})$. It is free over $\mathbb{Z}[\delta^{\pm 1}]$ of rank a_{2n} .*

Observe that R_n , $R_i R_{2n-i}$, E_n , and $E_i E_{2n-i}$ are fixed under σ for all $i \in \{1, \dots, n\}$. Thus the image of the map ϕ of Lemma 2.8 below lies in $\text{SBr}(A_{2n-1})$.

Lemma 2.8. *The following map determines a $\mathbb{Z}[\delta^{\pm 1}]$ -algebra homomorphism $\phi : \text{Br}(C_n) \rightarrow \text{SBr}(A_{2n-1})$.*

$$\begin{aligned} \phi(r_0) &= R_n, \quad \phi(r_i) = R_{n-i} R_{n+i}, \\ \phi(e_0) &= E_n \quad \text{and} \quad \phi(e_i) = E_{n-i} E_{n+i}, \quad \text{for } 0 < i < n. \end{aligned}$$

Proof. It suffices to verify that ϕ preserves the defining relations given in Definition 2.1. We demonstrate this for some of the relations (2.1)–(2.18), and leave the rest as an exercise for the reader.

For (2.13):

$$\begin{aligned} \phi(r_1)\phi(e_0)\phi(r_1)\phi(e_0) &= R_{n-1}(R_{n+1}E_nR_{n+1})R_{n-1}E_n \\ &\stackrel{(2.27)}{=} R_{n-1}R_nE_{n+1}(R_nR_{n-1}E_n) \\ &\stackrel{(2.26)}{=} R_{n-1}R_nE_{n+1}E_{n-1}E_n \\ &\stackrel{(2.24)+(2.26)}{=} E_nE_{n-1}E_{n+1}E_n \\ &= \phi(e_0)\phi(e_1)\phi(e_0). \end{aligned}$$

For (2.14):

$$\begin{aligned} \phi(r_1)\phi(r_0)\phi(r_1)\phi(e_0) &= R_{n+1}(R_{n-1}R_nR_{n-1})R_{n+1}E_n \\ &\stackrel{(2.25)}{=} R_{n+1}R_nR_{n-1}(R_nR_{n+1}E_n) \\ &\stackrel{(2.26)+(2.28)}{=} R_{n+1}R_nR_{n-1}E_{n+1}R_nR_{n+1} \\ &\stackrel{(2.23)}{=} R_{n+1}R_nE_{n+1}R_{n-1}R_nR_{n+1} \\ &\stackrel{(2.26)+(2.28)}{=} E_nR_{n+1}R_nR_{n-1}R_nR_{n+1} \\ &\stackrel{(2.25)}{=} E_nR_{n+1}R_{n-1}R_nR_{n-1}R_{n+1} \\ &= \phi(e_0)\phi(r_1)\phi(r_0)\phi(r_1). \end{aligned}$$

For (2.18):

$$\begin{aligned} \phi(e_1)\phi(e_0)\phi(r_1) &\stackrel{(2.22)}{=} E_{n-1}(E_{n+1}E_nR_{n+1})R_{n-1} \stackrel{(2.31)}{=} E_{n-1}E_{n+1}R_nR_{n-1} \\ &\stackrel{(2.24)+(2.28)}{=} E_{n+1}E_{n-1}E_n = \phi(e_1)\phi(e_0). \end{aligned}$$

□

At this point, we have explained the algebras and the map ϕ occurring in Theorem 1.1. The surjectivity of ϕ will be proved in Proposition 5.16 and its injectivity at the end of Section 6.

3 The classical Brauer algebra

Let $m \in \mathbb{N}$. In this section, we describe the root system of the Coxeter group of type A_m , focussing on special collections of mutually orthogonal positive roots called admissible sets. Also, the notion of height for elements of the Brauer monoid $\text{BrM}(A_m)$ is introduced and discussed. A major goal, established in Theorem 3.9, is to exhibit a normal form for elements of the monoid $\text{BrM}(A_m)$ as a product of generators.

Definition 3.1. Let $m \geq 1$. The root system of the Coxeter group $W(A_m)$ of type A_m is denoted by Φ . It is realized as $\Phi := \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq m+1, i \neq j\}$ in the Euclidean space \mathbb{R}^{m+1} , where ϵ_i is the i^{th} standard basis vector. Put $\alpha_i := \epsilon_i - \epsilon_{i+1}$. Then $\{\alpha_i\}_{i=1}^m$ is called the set of simple roots of Φ . Denote by Φ^+ the set of positive roots in Φ with respect to these simple roots; that is, $\Phi^+ := \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq m+1\}$.

We have seen that, up to powers of δ , the monomials of $\text{Br}(A_m)$ correspond to Brauer diagrams. In order to work with the tops and bottoms of Brauer diagrams, we introduce the following notion.

Definition 3.2. Let \mathcal{A} denote the collection of all subsets of Φ consisting of mutually orthogonal positive roots. Members of \mathcal{A} are called *admissible sets*.

An admissible set B corresponds to a Brauer diagram top in the following way: for each $\beta \in B$, where $\beta = \epsilon_i - \epsilon_j$ for some $i, j \in \{1, \dots, m+1\}$ with $i < j$, draw a horizontal strand in the corresponding Brauer diagram top from the dot $(i, 1)$ to the dot $(j, 1)$. All horizontal strands on the top are obtained this way, so there are precisely $|B|$ horizontal strands. The top of the Brauer diagram of Figure 2, corresponds to the admissible set $\{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_5 + \alpha_6 + \alpha_7\}$.

Similarly, there is an admissible set corresponding to a Brauer diagram bottom. The bottom of the Brauer diagram of Figure 2, corresponds to the admissible set $\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_4\}$.

For any $\beta \in \Phi^+$ and $i \in \{1, \dots, m\}$, there exists a $w \in W(A_m)$ such that $\beta = w\alpha_i$. Then $E_\beta := wE_iw^{-1}$ is well defined (see [4, Lemma 4.2]). If $\beta, \gamma \in \Phi^+$ are mutually orthogonal, then E_β and E_γ commute (see [4, Lemma 4.3]). Hence, for $B \in \mathcal{A}$, we can define the product

$$E_B = \prod_{\beta \in B} E_\beta, \quad (3.1)$$

which is a quasi-idempotent, and the normalized version

$$\hat{E}_B = \delta^{-|B|} E_B, \quad (3.2)$$

which is an idempotent element of the Brauer monoid.

There is an action of the Brauer monoid $\text{BrM}(A_m)$ on the collection \mathcal{A} . The generators R_i ($i = 1, \dots, m$) act by the natural action of Coxeter group elements on its root sets, where negative roots are negated so as to obtain positive roots, the element δ acts as the identity, and the action of E_i ($i = 1, \dots, m$) is defined by

$$E_i B := \begin{cases} B & \text{if } \alpha_i \in B, \\ B \cup \{\alpha_i\} & \text{if } \alpha_i \perp B, \\ R_\beta R_i B & \text{if } \beta \in B \setminus \alpha_i^\perp. \end{cases} \quad (3.3)$$

Alternatively, this action can be described as follows for a monomial a : complete the top corresponding to B into a Brauer diagram b , without increasing the number of horizontal strands in the top. Now aB is the top of the Brauer diagram ab . We will make use of this action in order to provide a normal form for elements of $\text{BrM}(A_m)$. In [4, Definition 3.2], it is shown that this action is well defined for any spherical simply laced type.

The action defined above is a left action. Similarly, there is a right action of $\text{BrM}(A_m)$ on \mathcal{A} . In order to interpret the right action of a on B , the latter should be pictured as the bottom of a Brauer diagram, so that the bottom corresponding to Ba is the bottom of ba . Observe that $a\emptyset$ is the top of the Brauer diagram of a (i.e., the collection of top horizontal strands of a and top row of points) and $\emptyset a$ is its bottom (i.e., the collection of bottom horizontal strands of a and bottom row of points).

Recall that the height $\text{ht}(\beta)$ of a positive root $\beta \in \Phi^+$ is h if it is the sum of precisely h simple roots. This definition will be extended to a height function on \mathcal{A} in such a way that $\text{ht}(\{\beta\}) = \text{ht}(\beta) - 1$.

Definition 3.3. The height of an admissible set B , notation $\text{ht}(B)$, is the minimal number of crossings in a completion of the top corresponding to B to a Brauer diagram without increasing the number of horizontal strands at the top.

For example, the height of the top of the Brauer diagram of Figure 2 is equal to 4 and the height of the bottom is equal to 3.

Definition 3.4. For every element $a \in \text{BrM}(A_m)$, we define the height of a , denoted by $\text{ht}(a)$, as the minimal number of generators R_i needed to write a as a product of the generators $R_1, \dots, R_m, E_1, \dots, E_m, \delta, \delta^{-1}$.

In terms of Brauer diagrams, the height of a is the minimal number of crossings needed to draw a . Consequently, the height of an admissible set B is the minimal height over all possible Brauer diagram completions of B .

The Brauer diagram of Figure 2 has height 7 if V is the identity and height 8 if $V = R_1$. The lemma below states some useful properties of this height function.

Remark 3.5. There is a natural anti-involution on $\text{Br}(A_m)$, denoted by $x \mapsto x^{\text{op}}$, determined by

$$R_i \mapsto R_i \text{ and } E_i \mapsto E_i.$$

By anti-involution, we mean a $\mathbb{Z}[\delta^{\pm 1}]$ -linear anti-automorphism whose square is the identity.

Lemma 3.6. *Let $B, C \in \mathcal{A}$ with $|B| = |C|$. Then there is a unique diagram $a_{B,C}$ of height $\text{ht}(B) + \text{ht}(C)$ in $\text{BrM}(A_m)$ such that $a_{B,C}\emptyset = B$ and $\emptyset a_{B,C} = C$. This diagram satisfies $a_{B,C}a_{B,C}^{\text{op}} = \delta^{|B|}E_B$ as well as $a_{B,C}C = B$ and $Ba_{B,C} = C$.*

Proof. The easiest proof to our knowledge is based on diagrams.

There is a unique way to complete a given top and bottom to a Brauer diagram with a minimal number of crossings: connect the first dot at the top from the left that is not the endpoint of a horizontal strand at the top to the first dot at the bottom that is not the endpoint of a horizontal strand at the bottom; proceed similarly with the second, and so on, until the Brauer diagram is complete. If the top corresponds to B and the bottom to C , the resulting diagram is the required monomial $a_{B,C}$. \square

Lemma 3.7. *Suppose $B \in \mathcal{A}$ has height 0. Then there are $r = m - 2|B|$ diagrams of height 1 in the group of invertible elements in $\hat{E}_B\text{BrM}(A_m)\hat{E}_B$ forming a Coxeter system of type A_r . (Here, invertibility is meant with respect to the unit \hat{E}_B of the monoid).*

Proof. As discussed above, a Brauer diagram with top and bottom corresponding to B has $r + 1$ free dots at the top and also $r + 1$ at the bottom. For a diagram to be an invertible element in $\hat{E}_B\text{BrM}(A_m)\hat{E}_B$, the remaining strands need to be vertical, so they belong to the symmetric group on the $r + 1$ free dots at the top (or those at the bottom). Now, up to powers of δ , the idempotent \hat{E}_B is the element in which all r vertical strands do not cross. Selecting diagrams in which the i^{th} and $(i + 1)^{\text{st}}$ vertical strands cross and no others (for $i = 1, 2, \dots, r$), we find the required Coxeter system of type A_r . \square

Definition 3.8. For $B \in \mathcal{A}$ of height 0, denote by K_B the Coxeter group determined by Lemma 3.7.

In the middle part of the right hand side of Figure 2, next to V , the element $e_3e_5e_7 = E_B$ appears, where $B = \{\alpha_3, \alpha_5, \alpha_7\}$ has height 0. Now K_B is a Coxeter group of type A_1 , generated by $R_1\hat{E}_B$, so the choices for V are consistent with the possibilities for $V\hat{E}_B = \hat{E}_BV\hat{E}_B \in K_B$.

Theorem 3.9. *Let $i \in \{0, 1, \dots, \lfloor m/2 \rfloor\}$ and let B be any admissible set of size i and of height 0. Then each element a of $\text{BrM}(\mathbf{A}_m)$ with $|a\emptyset| = i$ can be written uniquely as*

$$\delta^k UVW$$

for certain $k \in \mathbb{Z}$, U a diagram in $\text{BrM}(\mathbf{A}_m)E_B$ with $UB = a\emptyset$, W a diagram in $E_B\text{BrM}(\mathbf{A}_m)$ with $\emptyset a = BW$, and $V \in K_B$ such that

$$\text{ht}(a) = \text{ht}(U) + \text{ht}(V) + \text{ht}(W).$$

Proof. Take $U = a_{a\emptyset, B}$ and $W = a_{\emptyset a, B}^{\text{op}}$. Then $V = U^{\text{op}}aW^{\text{op}}$ has top and bottom equal to B and so belongs to K_B . By Lemma 3.6 and (3.2),

$$UVW = a_{a\emptyset, B}a_{a\emptyset, B}^{\text{op}}aa_{\emptyset a, B}a_{\emptyset a, B}^{\text{op}} = \delta^{4i}\hat{E}_{a\emptyset}a\hat{E}_{\emptyset a} = \delta^k a$$

for $k = 4i$. As $\text{ht}(U) = \text{ht}(a\emptyset)$ and $\text{ht}(W) = \text{ht}(\emptyset a)$ and $\text{ht}(V)$ is the length of V with respect to the Coxeter system of Lemma 3.7, this proves that a has a decomposition as stated.

As for uniqueness, suppose $a = UVW$ is a product decomposition as stated. Then $a\emptyset = UVB = UB = U\emptyset$ (as $U = U\hat{E}_B$) and $\emptyset U = B$, so by Lemma 3.7 and minimality of the height of U , we find $U = a_{a\emptyset, B}$. Similarly, W can be shown to be equal to $a_{\emptyset a, B}^{\text{op}}$. Finally, $V = \hat{E}_B V \hat{E}_B = \delta^{-4|B|}U^{\text{op}}UVWW^{\text{op}} = \delta^{-4|B|}U^{\text{op}}aW^{\text{op}}$ is uniquely determined by a , U , and W . \square

4 Elementary properties of type C algebras

In this section we draw some easy consequences from the definition of a Brauer algebra of type C_n . The results are of use in later sections.

Lemma 4.1. *In $\text{Br}(C_n, R, \delta)$, the following equations hold.*

$$r_1 e_0 e_1 = e_0 e_1 \tag{4.1}$$

$$e_0 e_1 e_0 = e_0 r_1 e_0 \tag{4.2}$$

$$e_1 r_0 r_1 e_0 = e_1 e_0 \tag{4.3}$$

$$r_0 r_1 e_0 r_1 = r_1 e_0 r_1 r_0 \tag{4.4}$$

$$e_0 r_1 e_0 r_1 = e_0 e_1 e_0 \tag{4.5}$$

Proof. By Definition 2.1,

$$r_1 e_0 e_1 \stackrel{(2.16)}{=} \delta^{-1} r_1 e_0 e_1 e_0 e_1 \stackrel{(2.13)}{=} \delta^{-1} e_0 e_1 e_0 e_1 \stackrel{(2.16)}{=} e_0 e_1,$$

proving (4.1). Therefore

$$e_0 e_1 e_0 \stackrel{(4.1)}{=} r_1 e_0 e_1 e_0 \stackrel{(2.13)}{=} r_1 r_1 e_0 r_1 e_0 \stackrel{(2.1)}{=} e_0 r_1 e_0$$

giving (4.2).

It is easy to check that (4.3) follows from (2.17) and (2.2). Also, the identity (4.4) holds as

$$r_0 r_1 e_0 r_1 \stackrel{(2.1)}{=} r_1 (r_1 r_0 r_1 e_0) r_1 \stackrel{(2.14)}{=} r_1 e_0 r_1 r_0 r_1 r_1 \stackrel{(2.1)}{=} r_1 e_0 r_1 r_0.$$

Finally,

$$(e_0 r_1 e_0) r_1 \stackrel{(4.2)}{=} e_0 (e_1 e_0 r_1) \stackrel{(2.18)}{=} e_0 e_1 e_0,$$

proving (4.5). \square

As in the case of type A_n (see Remark 3.5) there is similarly a natural anti-involution on $\text{Br}(C_n)$. This anti-involution is denoted by the superscript op , so the map is given by $x \mapsto x^{\text{op}}$.

Proposition 4.2. *The identity map on $\{\delta, r_i, e_i \mid i = 0, \dots, n-1\}$ extends to a unique anti-involution on the Brauer algebra $\text{Br}(C_n, R, \delta)$.*

Proof. It suffices to check the defining relations given in Definition 2.2 still hold under the anti-involution. An easy inspection shows that all relations involved in the definition are invariant under op , except for (2.9), (2.12), (2.13), (2.17), and (2.18). The relation obtained by applying op to (2.9) holds as can be seen by using (2.10) followed by (2.1) together with (2.9). The equality 2.17 is the op -dual of (2.12). Finally, (4.1) and (4.5) state that the op duals of (2.18) and (2.13), respectively, hold. \square

For each $i \in \{1, \dots, n\}$, we define the following two elements of $\text{BrM}(C_n)$.

$$y_i := r_{i-1} r_{i-2} \cdots r_1 r_0 r_1 \cdots r_{i-2} r_{i-1}, \quad (4.6)$$

$$z_i := r_{i-1} r_{i-2} \cdots r_1 e_0 r_1 \cdots r_{i-2} r_{i-1}. \quad (4.7)$$

Proposition 4.3. *Let $n \geq 2$ and $i \in \{2, \dots, n\}$ and consider elements in $\text{BrM}(C_n)$.*

(i) e_i, r_i, y_i , and z_i commute with each of r_j and e_j for $0 \leq j \leq i-2$.

(ii) y_i and z_i commute with y_j and z_j for each $j \in \{1, \dots, n\}$.

Proof. (i). By Definition 2.1, both e_i and r_i commute with each element of $\{r_0, \dots, r_{i-2}, e_0, \dots, e_{i-2}\}$.

In order to prove that y_i commutes with the indicated elements, we first establish the claim that y_{i+2} commutes with r_i and e_i , for $0 \leq i \leq n-2$.

If $i = 0$, the claim follows from (2.11) and (2.14), respectively. If $i > 0$, we have

$$\begin{aligned}
y_{i+2}r_i &= r_{i+1}r_i \cdots r_1r_0r_1 \cdots r_i r_{i+1}r_i \stackrel{(2.8)}{=} r_{i+1}r_i \cdots r_1r_0r_1 \cdots r_{i-1}r_{i+1}r_i r_{i+1} \\
&\stackrel{(2.5)}{=} r_{i+1}r_i r_{i+1}r_{i-1} \cdots r_1r_0r_1 \cdots r_i r_{i+1} \\
&\stackrel{(2.8)}{=} r_i r_{i+1}r_i \cdots r_1r_0r_1 \cdots r_{i-1}r_i r_{i+1} = r_i y_{i+2},
\end{aligned}$$

and

$$\begin{aligned}
y_{i+2}e_i &= r_{i+1}r_i \cdots r_1r_0r_1 \cdots r_i r_{i+1}e_i \stackrel{(2.10)}{=} r_{i+1}r_i \cdots r_1r_0r_1 \cdots r_{i-1}e_{i+1}r_i r_{i+1} \\
&\stackrel{(2.6)}{=} r_{i+1}r_i e_{i+1}r_{i-1} \cdots r_1r_0r_1 \cdots r_i r_{i+1} \\
&\stackrel{(2.10)}{=} e_i r_{i+1}r_i \cdots r_1r_0r_1 \cdots r_{i-1}r_i r_{i+1} = e_i y_{i+2}.
\end{aligned}$$

Now, for arbitrary i and all $0 \leq j \leq i-2$, using $y_i = r_{i-1} \cdots r_{j+2}y_{j+2}r_{j+2} \cdots r_{i-1}$ and (2.5), we find

$$\begin{aligned}
y_i r_j &= r_{i-1} \cdots r_{j+2}y_{j+2}r_{j+2} \cdots r_{i-1}r_j = r_{i-1} \cdots r_{j+2}y_{j+2}r_j r_{j+2} \cdots r_{i-1} \\
&= r_{i-1} \cdots r_{j+2}r_j y_{j+2}r_{j+2} \cdots r_{i-1} = r_j r_{i-1} \cdots r_{j+2}y_{j+2}r_{j+2} \cdots r_{i-1} \\
&= r_j y_i
\end{aligned}$$

Similarly for e_j instead of r_j , by use of (2.6).

An analogous argument can be used for z_i . Again, it suffices to show that z_{i+2} commutes with r_i and e_i . For $i = 0$ we verify

$$\begin{aligned}
z_2 e_0 &= (r_1 e_0 r_1) e_0 \stackrel{(2.13)+(4.5)}{=} e_0 (r_1 e_0 r_1) = e_0 z_2, \\
z_2 r_0 &= (r_1 e_0 r_1) r_0 \stackrel{(2.1)}{=} r_1 e_0 r_1 r_0 r_1 r_1 \stackrel{(2.14)+(2.1)}{=} r_0 (r_1 e_0 r_1) = e_0 z_2.
\end{aligned}$$

For $i > 0$, it is straightforward to show that $z_{i+2}r_i = r_i z_{i+2}$, using (2.5) and (2.8), and $z_{i+2}e_i = e_i z_{i+2}$, by using a rearrangement of (2.10). Thus, an argument similar to the above proves that $z_i r_j = r_j z_i$ and $z_i e_j = e_j z_i$, for any i and all $0 \leq j \leq i-2$.

As y_i and z_i are conjugates of r_0 and e_0 by the same Coxeter group element, it follows from (2.2) that they commute. It remains to verify that y_i and z_i commute with y_{i-1} and z_{i-1} . But the latter two elements are products of generators from $\{r_0, \dots, r_{i-2}, e_0, \dots, e_{i-2}\}$, which are known to commute with y_i and z_i by (i) and (ii). This finishes the proof of (i) and (ii). \square

Remark 4.4. Using Proposition 4.3, one can prove that for any $n \geq 1$,

$$\text{BrM}(C_n) = \text{BrM}(C_{n-1})\{1, e_{n-1}, r_{n-1}, y_n, z_n\}\text{BrM}(C_{n-1}).$$

This ensures that $\text{Br}(C_n)$ is of *finite rank* over $\mathbb{Z}[\delta^{\pm 1}]$. Details of the proof of the ensuing ‘normal form’ are suppressed here, as this result is not used in this paper.

5 Surjectivity of ϕ

The goal of this section is to exhibit a collection of admissible sets on which $\text{BrM}(C_n)$ acts as well as to prove that the map $\phi : \text{Br}(C_n) \rightarrow \text{SBr}(A_{2n-1})$ introduced in Theorem 1.1 is surjective. To this end, we first construct the root system of type C_n in terms of σ -fixed vectors in the reflection space for $W(A_{2n-1})$ spanned by the root system Φ of Definition 3.1. Notice that the restriction of ϕ to the submonoid $W(C_n)$ of $\text{BrM}(C_n)$ generated by the r_i (isomorphic, as the notation suggests, to the Coxeter group of type C_n) is known to be injective (see, for instance [11]), with image the centralizer of σ in the submonoid $W(A_{2n-1})$ of $\text{BrM}(A_{2n-1})$.

We adopt the notation of Section 3 with $m = 2n - 1$, and will use the root system Φ , the collection of admissible sets \mathcal{A} , and the action of $\text{BrM}(A_{2n-1})$ on \mathcal{A} defined there. We let the involution σ act on the set Φ^+ of positive roots of Φ in the following way, where α_i are as in Definition 3.1. For $\Sigma c_i \alpha_i \in \Phi^+$ we decree

$$\sigma(\Sigma c_i \alpha_i) = \Sigma c_i \alpha_{2n-i} \quad (1 \leq i < 2n).$$

This map σ induces the permutation of the simple roots corresponding to the nontrivial automorphism of the Coxeter diagram A_{2n-1} . This permutation can be extended to the linear transformation of \mathbb{R}^{2n} , again denoted σ , determined by $\sigma(\epsilon_i) = -\epsilon_{2n+1-i}$. (Indeed, this transformation satisfies $\sigma(\alpha_i) = \alpha_{2n-i}$ for each $i \in \{1, \dots, 2n-1\}$). The vectors fixed by σ form an n -dimensional subspace, to be denoted \mathbb{R}_σ^{2n} , of the $(2n-1)$ -dimensional subspace of \mathbb{R}^{2n} spanned by Φ .

Definition 5.1. Let $\mathbf{p} : \mathbb{R}^{2n} \rightarrow \mathbb{R}_\sigma^{2n}$ be the orthogonal projection from \mathbb{R}^{2n} onto \mathbb{R}_σ^{2n} , that is, $\mathbf{p}(x) = (x + \sigma(x))/2$ for $x \in \mathbb{R}^{2n}$. Let $\alpha \in \Phi$. Then $\mathbf{p}(\alpha) = \alpha$ is of squared norm 2 if $\sigma(\alpha) = \alpha$ and $\mathbf{p}(\alpha) = \frac{1}{2}(\alpha + \sigma(\alpha))$ is of squared norm 1 if $\sigma(\alpha) \neq \alpha$. The image $\Psi = \mathbf{p}(\Phi)$ of Φ under \mathbf{p} is a root system of type C_n with simple roots $\beta_0 = \mathbf{p}(\alpha_n) = \alpha_n$ and $\beta_i = \mathbf{p}(\alpha_{n-i}) = \mathbf{p}(\alpha_{n+i})$ for $i = 1, \dots, n-1$. It is contained in \mathbb{R}_σ^{2n} and spans it. Of course, Ψ^+ will be understood to be the half of Ψ lying in the cone spanned by β_i ($i = 0, \dots, n-1$). Given $\alpha \in \Phi$ we write R_α for the orthogonal reflection on \mathbb{R}^{2n} with root α . Given $\beta \in \Psi$ we write r_β for the orthogonal reflection on \mathbb{R}_σ^{2n} with root β . We may identify R_i and r_j with R_{α_i} and r_{β_j} , respectively.

Recall that $\phi(r_0) = R_n$ and $\phi(r_j) = R_{n-j}R_{n+j}$ for $j \in \{1, \dots, n-1\}$.

Lemma 5.2. *The map $\sigma : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ and the restriction of ϕ to $W(C_n)$ satisfy the following properties for each $w \in W(C_n)$.*

- (i) $\sigma\phi(w) = \phi(w)\sigma$.
- (ii) $\phi(w)x = wx$ if $x \in \mathbb{R}_\sigma^{2n}$.

Proof. As $\sigma R_i \sigma^{-1} = R_{2n-i}$ for each $i \in \{1, \dots, 2n-1\}$, we know that $\sigma R_n \sigma^{-1} = R_n$, and $\sigma R_{n+j} R_{n-j} \sigma^{-1} = R_{n-j} R_{n+j}$, so σ commutes with $\phi(r_j)$ for each $j \in \{0, \dots, n-1\}$. As r_0, \dots, r_{n-1} generate $W(C_n)$, this implies that σ commutes with each $\phi(w)$ for $w \in W(C_n)$. Hence (i).

Also for (ii) it suffices to verify the statement for w a simple reflection. Let $x \in \mathbb{R}_\sigma^{2n}$. For $j = 0$, we have $\phi(r_0)x = R_n x = r_0 x$ and for $j = 1, \dots, n-1$, as $(x, \alpha_{n-j}) = (x, \alpha_{n+j})$,

$$\begin{aligned} \phi(r_j)x &= R_{n-j} R_{n+j} x = R_{n-j} (x - (x, \alpha_{n+j}) \alpha_{n+j}) \\ &= x - (x, \alpha_{n+j}) \alpha_{n+j} - (x, \alpha_{n-j}) \alpha_{n-j} \\ &= x - (x, \alpha_{n+j} + \alpha_{n-j}) (\alpha_{n+j} + \alpha_{n-j}) / 2 \\ &= r_j x, \end{aligned}$$

as required. \square

The following result is well known and gives the connection between the restriction of ϕ to $W(C_n)$ and \mathfrak{p} .

Lemma 5.3. *The maps \mathfrak{p} and ϕ are compatible in the following two ways.*

- (i) $\mathfrak{p}(\phi(w)\alpha) = w\mathfrak{p}(\alpha)$ for each $w \in W(C_n)$ and $\alpha \in \Phi$.
- (ii) $\phi(r_\beta) = \prod_{\alpha \in \mathfrak{p}^{-1}(\beta)} R_\alpha$ and $\phi(e_\beta) = \prod_{\alpha \in \mathfrak{p}^{-1}(\beta)} E_\alpha$ for each $\beta \in \Psi$. Here, the set $\mathfrak{p}^{-1}(\beta)$ has cardinality 1 or 2, according to $\beta \in W(C_n)\beta_0$ or $\beta \in W(C_n)\beta_1$.

Proof. By the definition of \mathfrak{p} and Lemma 5.2,

$$\begin{aligned} \mathfrak{p}(\phi(w)\alpha) &= (\phi(w)\alpha + \sigma(\phi(w)\alpha)) / 2 = (\phi(w)\alpha + \phi(w)\sigma(\alpha)) / 2 \\ &= \phi(w)(\alpha + \sigma(\alpha)) / 2 = \phi(w)(\mathfrak{p}(\alpha)) \\ &= w(\mathfrak{p}(\alpha)), \end{aligned}$$

which proves (i).

As for (ii), let $\beta \in \Psi$. The proofs for r_β and e_β are almost identical, so we only give the former. If β is simple, the equality $\phi(r_\beta) = \prod_{\alpha \in \mathfrak{p}^{-1}(\beta)} R_\alpha$ holds by definition of ϕ . Suppose $\beta = w\beta_0$ for some $w \in W(C_n)$. Then $\beta = \phi(w)\beta_0$ by Lemma 5.2(ii) and so, by (i) of the same lemma,

$$\mathfrak{p}(\beta) = \mathfrak{p}(\phi(w)\beta_0) = w\mathfrak{p}(\beta_0) = w\beta_0 = \beta.$$

Now $\mathfrak{p}^{-1}(\beta) = \{\beta\}$ and

$$\begin{aligned} \phi(r_\beta) &= \phi(wr_0w^{-1}) = \phi(w)\phi(r_0)\phi(w)^{-1} = \phi(w)R_n\phi(w)^{-1} \\ &= R_{\phi(w)\alpha_n} = R_{w\alpha_n} = R_\beta = \prod_{\alpha \in \mathfrak{p}^{-1}\beta} R_\alpha. \end{aligned}$$

As Ψ is the union of the two $W(C_n)$ -orbits with representatives β_0 and β_1 , it only remains to consider $\beta = w\beta_1$ with $w \in W(C_n)$. Take $\alpha \in \mathfrak{p}^{-1}(\beta)$. Then $\mathfrak{p}(\phi(w)\alpha_{n-1}) = w\mathfrak{p}(\alpha_{n-1}) = w\beta_1 = \beta = \mathfrak{p}(\alpha)$, so, in view of Lemma 5.2(i), $\mathfrak{p}^{-1}(\beta) = \{\phi(w)\alpha_{n-1}, \phi(w)\alpha_{n+1}\}$. We find

$$\begin{aligned} \phi(r_\beta) &= \phi(wr_1w^{-1}) = \phi(w)\phi(r_1)\phi(w)^{-1} = \phi(w)R_{n-1}R_{n+1}\phi(w)^{-1} \\ &= R_{\phi(w)\alpha_{n-1}}R_{\phi(w)\alpha_{n+1}} = \prod_{\alpha \in \mathfrak{p}^{-1}(\beta)} R_\alpha, \end{aligned}$$

which establishes (ii). \square

We next consider particular sets of mutually orthogonal positive roots in Ψ , and relate them to symmetric admissible sets in \mathcal{A} .

Definition 5.4. Denote by \mathcal{B}' the collection of all sets of mutually orthogonal roots in Ψ^+ and by \mathcal{A}_σ the subset of σ -invariant elements of \mathcal{A} . As \mathfrak{p} sends positive roots of Φ to positive roots of Ψ , it induces a map $\mathfrak{p} : \mathcal{A}_\sigma \rightarrow \mathcal{B}'$ given by $\mathfrak{p}(B) = \{\mathfrak{p}(\alpha) \mid \alpha \in B\}$ for $B \in \mathcal{A}_\sigma$. An element of \mathcal{B}' will be called *admissible* if it lies in the image of \mathfrak{p} . The set of all admissible elements of \mathcal{B}' will be denoted \mathcal{B} .

Remark 5.5. Not all sets of mutually orthogonal roots in Ψ^+ are admissible. For instance $Y = \{\beta_1, \beta_1 + \beta_0\}$ (two mutually orthogonal short roots) belongs to \mathcal{B}' (for $n = 2$) but not to \mathcal{B} . For, if $X \in \mathcal{A}_\sigma$ would be such that $\mathfrak{p}(X) = Y$, then X should contain α_1 and α_3 as well as $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$; but these roots are not mutually orthogonal. On the other hand, for $n \geq 4$, the unordered pair $\{\beta_1, \beta_3\}$ from another $W(C_n)$ -orbit of mutually orthogonal short roots, is the image of the admissible set $\{\alpha_{n-1}, \alpha_{n+1}, \alpha_{n-3}, \alpha_{n+3}\}$ and so belongs to \mathcal{B} .

Also $\{\beta_0, 2\beta_1 + \beta_0\}$ (two mutually orthogonal long roots) does belong to \mathcal{B} (for $n = 2$) as it coincides with $\mathfrak{p}(X)$, where $X = \{\alpha_n, \alpha_{n-1} + \alpha_n + \alpha_{n+1}\}$.

Proposition 5.6. *The monoid $\text{BrM}(C_n)$ acts on \mathcal{A}_σ under the composition of the above action and ϕ .*

Proof. It suffices to prove that \mathcal{A}_σ is closed under the action of $\phi(\text{BrM}(C_n))$. It is easy to see that $\sigma(a)\sigma(B) = \sigma(aB)$, for $a \in \text{BrM}(A_{2n-1})$ and $B \in \mathcal{A}$. Consequently, if $a \in \text{BrM}(A_{2n-1})_\sigma$ and $B \in \mathcal{A}_\sigma$, then it follows that $aB = \sigma(a)\sigma(B) = \sigma(aB)$. This shows $aB \in \mathcal{A}_\sigma$. As $\phi(\text{BrM}(C_n)) \subseteq \text{BrM}(A_{2n-1})_\sigma$, the proposition follows. \square

Proposition 5.7. *The map $\mathfrak{p} : \mathcal{A}_\sigma \rightarrow \mathcal{B}$ is bijective and $W(C_n)$ -equivariant, so $\mathfrak{p}(\phi(w)X) = w\mathfrak{p}(X)$ for $X \in \mathcal{A}_\sigma$ and $w \in W(C_n)$.*

Proof. The map \mathbf{p} is surjective by definition of \mathcal{B} . Let $Y \in \mathcal{B}$ and $X \in \mathbf{p}^{-1}(Y)$. If $\beta \in Y$, then there is $\alpha \in X$ such that $\beta = \mathbf{p}(\alpha)$. As $X \in \mathcal{A}_\sigma$, it follows that $\sigma\alpha \in X$, so $X = \{\alpha \in \Phi \mid \mathbf{p}(\alpha) \in Y\}$ is uniquely determined by Y . This shows that \mathbf{p} is injective.

Finally, if in addition, $w \in W(C_n)$, then $\phi(w)X \in \mathcal{A}_\sigma$ by Proposition 5.6, and, by Lemma 5.3,

$$\mathbf{p}(\phi(w)X) = \{\mathbf{p}(\phi(w)\alpha) \mid \alpha \in X\} = w\{\mathbf{p}(\alpha) \mid \alpha \in X\} = w\mathbf{p}(X).$$

□

Lemma 5.8. *Let i and j be nodes of the Dynkin diagram C_n . If $w \in W(C_n)$ satisfies $w\beta_i = \beta_j$, then $we_iw^{-1} = e_j$.*

Proof. Observe that $w\beta_i = \beta_j$ only holds for distinct i and j if $i, j > 0$, in which case the existence of such a w is a direct consequence of known results on Coxeter groups. The full statement then follows from the fact that all generators of the stabilizer of β_i in $W(C_n)$ also stabilize e_i , which we prove now.

Suppose $i > 0$. As

$$\begin{aligned} r_1 e_1 r_1 & \stackrel{(2.2)}{=} e_1, \\ r_0(r_1 r_0 e_1 r_0 r_1) r_0 & \stackrel{(2.12) + (2.17)}{=} r_0 r_0 e_1 r_0 r_0 = e_1, \\ y_3 e_1 y_3 & \stackrel{4.3}{=} e_1 y_3 y_3 = e_1, \\ r_i e_1 r_i & \stackrel{(2.7)}{=} e_1 r_i r_i = e_1, \text{ for } i > 3, \end{aligned}$$

the elements $r_1, r_0 r_1 r_0, y_3, r_3, \dots, r_{n-1}$ stabilize e_1 . But these elements are known to generate the full stabilizer in $W(C_n)$ of β_1 , so $w\beta_1 = \beta_1$ implies $we_1 w^{-1} = e_1$ for every $w \in W(C_n)$.

If one of i and j is 0, then they both are, as $(we_i w^{-1})^2 = \delta^k we_i w^{-1}$, where $k = 2$ if $i > 0$ and $k = 1$ otherwise (see (2.3) and (2.4)).

It is known that the stabilizer of β_0 is generated by $r_0, r_1 r_0 r_1$, and r_i ($i = 2, \dots, n-1$). These elements also centralize e_0 :

$$\begin{aligned} r_0 e_0 r_0 & \stackrel{(2.2)}{=} e_0, \\ r_1 r_0 r_1 e_0 r_1 r_0 r_1 & \stackrel{(2.14)}{=} e_0, \\ r_i e_0 r_i & \stackrel{(2.6)}{=} e_0 \quad \text{for } i > 1. \end{aligned}$$

This ends the proof of the lemma. □

Consider a positive root β and a node i of type C_n . If there exists $w \in W$ such that $w\beta_i = \beta$, then we can define the element e_β in $\text{BrM}(C_n)$ by

$$e_\beta = we_iw^{-1}.$$

The above lemma implies that e_β is well defined. In general,

$$we_\beta w^{-1} = e_{w\beta},$$

for $w \in W(C_n)$ and β a root of $W(C_n)$. Note that $e_\beta = e_{-\beta}$ in view of (2.2).

In this perspective, we can reinterpret the element y_i of (4.6) as r_γ and z_i of (4.7) as e_γ , where $\gamma = \beta_0 + 2\beta_1 + \cdots + 2\beta_{i-1}$. Proposition 4.3 shows that, for each $i \in \{0, \dots, n-1\}$,

$$b_i = \prod_{k=1}^i z_k \quad (5.1)$$

is well defined. The admissible set

$$B_i = \mathfrak{p}^{-1}(\{\beta_0, \beta_0 + 2\beta_1, \dots, \beta_0 + 2\beta_1 + \cdots + 2\beta_{i-1}\}) \quad (5.2)$$

in \mathcal{A}_σ is both the top and the bottom of $\phi(b_i)$. In other words, the symmetric diagram of b_i has horizontal strands from $(n+1-j, 1)$ to $(n+j, 1)$ and from $(n+1-j, 0)$ to $(n+j, 0)$ for each $j \in \{1, \dots, i\}$. This is the special case $p = i$ of the top displayed in Figure 3. Later, below (5.5), we will use this fact. The B_i ($i = 0, \dots, n$) are a complete set of $W(A_m)$ -orbit representatives in \mathcal{A} . Moreover, $\phi(b_i)$ has height 0.

Proposition 5.9. *Let β and γ be positive roots of Ψ .*

- (i) $e_\beta r_\beta = r_\beta e_\beta = e_\beta$, $e_\beta^2 = \delta^2 e_\beta$ if β is short, and $e_\beta^2 = \delta e_\beta$ if β is long.
- (ii) If $(\beta, \gamma) = \pm 1$ and β and γ are short, then $e_\beta r_\gamma e_\beta = e_\beta$, $r_\beta r_\gamma e_\beta = e_\gamma r_\beta r_\gamma = e_\gamma e_\beta$, and $e_\beta e_\gamma e_\beta = e_\beta$.
- (iii) If $(\beta, \gamma) = \pm 1$ with β short and γ long, then the equations (2.11)–(2.18) and (4.1)–(4.5) still hold with the subscripts 1 and 0 replaced by β and γ , respectively.
- (iv) If $(\beta, \gamma) = 0$ and β and γ are both long, then $e_\beta e_\gamma = e_\beta e_\gamma$.
- (v) If $(\beta, \gamma) = 0$ and β and γ are both short, and there exists a long positive root α such that $\beta = \gamma + \alpha$ or $\beta = \gamma - \alpha$, then $e_\beta e_\gamma = \delta r_\alpha e_\gamma$ or $e_\gamma e_\beta = \delta r_\alpha e_\beta$. In each case, $e_\beta e_\gamma \neq e_\gamma e_\beta$.

Proof. The assertions are easily proved after reduction to simple cases using Lemma 5.8. We illustrate the argument by treating (v) now in greater detail.

Up to an interchange of β and γ , there exists $w \in W(C_n)$ such that $w\beta_0 = \alpha$ and $w\beta_1 = \gamma$ and $\beta = \gamma + \alpha = r_\alpha\gamma$. Now $e_\beta e_\gamma = r_\alpha e_\gamma r_\alpha e_\gamma = wr_0 e_1 r_0 e_1 w^{-1} \stackrel{(2.15)}{=} \delta w r_0 e_1 w^{-1} = \delta r_\alpha e_\gamma$. The inequality stated at the end of part (v) follows from the inspection of symmetric Brauer diagrams in the image of ϕ . \square

As a consequence of Proposition 5.9(iv) and (2.7), the product of e_β , for β running over the members of an admissible set, does not depend on the order. Therefore, for each $B \in \mathcal{B}$, we may define

$$e_B = \prod_{\beta \in B} e_\beta, \quad (5.3)$$

This is very similar to the definition of E_B in (3.2). Part (v) of the proposition shows that it is essential that B be admissible for a set B of orthogonal roots to define a product as in (5.3). There exists another $W(C_n)$ -orbit of pairs of mutually orthogonal positive roots than those in part (v) of the previous proposition; these lead to admissible sets and behave well in (5.3) in view of (2.7).

As $W(C_n)$ is a subgroup of the monoid $\text{BrM}(C_n)$, it also acts on \mathcal{A}_n (cf. Proposition 5.6). We will show that the admissible sets defined below are orbit representatives for this action.

Definition 5.10. For i and p with $0 \leq p \leq i \leq n$ and $i - p$ even, write

$$e_{i,p} = e_{p+1} e_{p+3} \cdots e_{i-1}, \quad (5.4)$$

and

$$B_{i,p} = e_{p+1} e_{p+3} \cdots e_{i-1} B_i.$$

In addition, for $p' \in \mathbb{N}$ with $0 \leq p' \leq i < n$ and $i - p'$ even, we write

$$b_{p,i,p'} = e_{i,p} b_i e_{i,p'}, \quad (5.5)$$

where b_i is as defined in (5.1).

Observe that $B_{i,i} = B_i$. The admissible set $B_{i,p}$ is pictured in Figure 3 as the top of a Brauer diagram.

Lemma 5.11. *If $0 < p, p' < i$ with $i - p$ and $i - p'$ even, then $\phi(b_{p,i,p'})$ is a diagram of height 0 with top $B_{i,p}$ and bottom $B_{i,p'}$. Moreover, $\phi(b_{p,i,p'}) = a_{B_{i,p}, B_{i,p'}}$.*

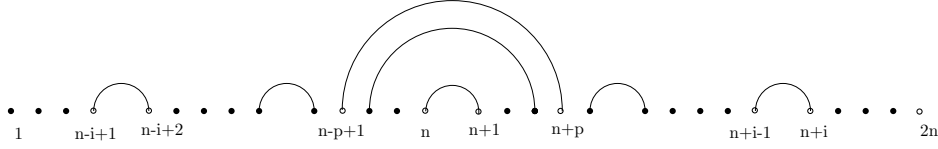


Figure 3: The admissible set $B_{i,p}$.

Proof. This is an easy verification involving the left and right monoid actions of Proposition 5.6 (use that $e_{i,p'}^{\text{op}} = e_{i,p'}$). \square

Recall that the Temperley–Lieb subalgebra of $\text{Br}(\mathbb{C}_n)$ is the subalgebra generated by e_0, \dots, e_{n-1} , and similarly for $\text{Br}(\mathbb{A}_m)$ and for the corresponding monoids. The following lemma implies that the restriction of ϕ to the Temperley–Lieb subalgebra of $\text{Br}(\mathbb{C}_n)$ behaves as required.

Lemma 5.12. *Let $B \in \mathcal{A}_\sigma$. Then $E_B = \phi(e_{\mathfrak{p}(B)})$ and, in particular, hence $E_B \in \phi(\text{Br}(\mathbb{C}_n))$. Moreover, the restriction of ϕ to the Temperley–Lieb subalgebra of $\text{Br}(\mathbb{C}_n)$ is surjective onto the intersection of the Temperley–Lieb algebra of $\text{Br}(\mathbb{A}_{2n-1})$ with $\text{SBr}(\mathbb{A}_{2n-1})$.*

Proof. Suppose $B \in \mathcal{A}_\sigma$. Then, by Lemmas 5.3(ii) and 2.8,

$$E_B = \prod_{\alpha \in B} E_\alpha = \prod_{\beta \in \mathfrak{p}(B)} E_{\mathfrak{p}^{-1}(\beta)} = \prod_{\beta \in \mathfrak{p}(B)} \phi(e_\beta) = \phi \left(\prod_{\beta \in \mathfrak{p}(B)} e_\beta \right) = \phi(e_{\mathfrak{p}(B)}),$$

as required for the first statements.

Let $\bar{i} \in \{0, 1\}$ be such that $i \equiv \bar{i} \pmod{2}$. Then $B_{i,\bar{i}}$ consists of simple roots only and belongs to the same $W(\mathbb{A}_{2n-1})$ -orbit in \mathcal{A} as B_i or in fact $B_{i,p}$ for any p .

As for the last statement, suppose that $a \in \text{SBr}(\mathbb{A}_{2n-1})$ is a monomial in the Temperley–Lieb subalgebra of $\text{Br}(\mathbb{A}_{2n-1})$. Then, up to powers of δ , there are $i \in \{0, \dots, n\}$ and $B, B' \in \mathcal{A}_\sigma$ of height 0 with $a = a_{B,B'} = (a_{B,B_{i,\bar{i}}})(a_{B',B_{i,\bar{i}}})^{\text{op}}$ in the notation of and by use of Lemma 3.6. In view of the opposition involution and the first statement of this proposition, it suffices to show that $a := a_{B,B_{i,\bar{i}}}$ lies in the image under ϕ of the submonoid of $\text{BrM}(\mathbb{C}_n)$ generated by e_0, e_1, \dots, e_{n-1} , the Temperley–Lieb submonoid of $\text{BrM}(\mathbb{C}_n)$. To this end, let $B \in \mathcal{A}_\sigma$ be of height 0. We will establish the existence of an element b in this Temperley–Lieb submonoid with $\phi(b)B_{i,\bar{i}} = B$. This will suffice as then Theorem 3.9 gives $a = E_B \phi(b) = \phi(e_{\mathfrak{p}(B)} b)$, up to powers of δ , and so $a \in \phi(\text{BrM}(\mathbb{C}_n))$.

For $\gamma_1 = \epsilon_{i_1} - \epsilon_{j_1}$ and $\gamma_2 = \epsilon_{i_2} - \epsilon_{j_2} \in \Phi^+$, we say $\gamma_1 \ll \gamma_2$ if $1 \leq i_2 < i_1 < j_1 < j_2 \leq 2n$. If $\gamma_1 \ll \gamma_2 \ll \dots \ll \gamma_s$, and $\gamma_k \in B$, for $1 \leq k \leq s$, we

say that $\gamma_1 \ll \gamma_2 \ll \cdots \ll \gamma_s$ is a *chain in B* , and call s the *length* of this chain. Now we use induction on the maximal length t of a chain in B and the number l of longest chains in B .

Suppose first $t = 1$. Then B consists of simple roots and b as required can be found easily. Suppose $t > 1$. Let $\gamma_1 \ll \gamma_2 \ll \cdots \ll \gamma_t$ be a longest chain of B with $r_2 = \epsilon_{j_2} - \epsilon_{i_2}$. Since a is an element of the Temperley–Lieb submonoid of $\text{BrM}(A_{2n-1})$, $j_2 - i_2$ is odd, and for $1 \leq k \leq r = [(j_2 - i_2)/2]$ we have $\epsilon_{i_2+2k-1} - \epsilon_{i_2+2k} \in B$. Let

$$\begin{aligned} D &= \{\gamma_2\} \cup \{\epsilon_{i_2+2k-1} - \epsilon_{i_2+2k}\}_{k=1}^r, \\ D' &= \{\epsilon_{i_2+2k} - \epsilon_{i_2+2k+1}\}_{k=0}^r, \\ B' &= (B \setminus D \cup \sigma(D)) \cup (D' \cup \sigma(D')). \end{aligned}$$

Now $B' \in \mathcal{A}_\sigma$ is the top of a Temperley–Lieb element with maximal length less than t or number of maximal chains fewer than l . By the induction hypothesis there exists some Temperley–Lieb element b' in $\text{BrM}(C_n)$ with $\phi(b')B_i = B'$. It satisfies

$$E_X(D' \cup \sigma(D')) = D \cup \sigma(D) \quad \text{and} \quad E_X(B') = B,$$

where $X = (D \setminus \{\gamma_2\}) \cup \sigma(D \setminus \{\gamma_2\}) \in \mathcal{A}_\sigma$. As $E_X = \phi(e_{\mathfrak{p}(X)})$ by the first statement of the lemma, we conclude that $\phi(e_{\mathfrak{p}(X)}b')B_{i,\bar{i}} = B$, which proves the lemma. \square

Definition 5.13. Let $i \in \{0, \dots, n\}$. Define $\hat{E}^{(i)}$ to be the idempotent in $\text{BrM}(A_{2n-1})$ corresponding to b_i ;

$$\hat{E}^{(i)} := \delta^{-i} \phi(b_i).$$

In addition, define

$$\hat{b}^{(i)} := \delta^{-i} e_{\mathfrak{p}(B_i)}. \quad (5.6)$$

Then $\hat{E}^{(i)} = \phi(\hat{b}^{(i)})$ by Lemma 5.12. Furthermore, we write K_i instead of K_{B_i} as introduced in Definition 3.8.

Observe that K_i is generated by

$$\hat{E}^{(i)} R_{n \pm [2+i/2]}, \hat{E}^{(i)} R_{n \pm [3+i/2]}, \dots, \hat{E}^{(i)} R_{n \pm (n-1)}, \text{ and } \hat{E}^{(i)} R_0, \quad (5.7)$$

where R_0 stands for the longest reflection of $W(A_{2n-1})$, that is,

$$(R_1 R_{2n-1})(R_2 R_{2n-2}) \cdots (R_{n-1} R_{n+1}) R_n (R_{n-1} R_{n+1}) \cdots (R_2 R_{2n-2})(R_1 R_{2n-1}).$$

The definition of y_i from (4.6) shows that $\phi(y_n) = R_0$.

We will now prove that the σ -fixed part of K_i is contained in the image of ϕ , making use of the fact that K_i is a Coxeter group of which the above generators are a Coxeter system, as given by Lemma 3.7.

Lemma 5.14. *Let $i \in \{0, \dots, n\}$. The set (5.7) of simple reflections of K_i is invariant under σ . In fact, σ induces the nontrivial automorphism on the Coxeter type $A_{2(n-j)-1}$ of K_i , where $j = 1 + \lfloor i/2 \rfloor$. As a consequence, the subgroup of σ -fixed elements of K_i is of type C_{n-j} and is generated by the images under ϕ of $\hat{b}^{(i)}r_{j+1}, \hat{b}^{(i)}r_{j+2}, \dots, \hat{b}^{(i)}r_{n-1}$, and $\hat{b}^{(i)}y_n\hat{b}^{(i)}$.*

Proof. Clearly, σ fixes $\hat{E}^{(i)} = \phi(\hat{b}^{(i)})$. Moreover, it fixes R_0 and interchanges R_{n-k} and R_{n+k} , so indeed the Coxeter system of K_i is σ -invariant and σ induces the nontrivial automorphism on the Coxeter type $A_{2(n-j)-1}$ of K_i . It is well known (cf. [11]) that the subgroup of σ -fixed elements of K_i is generated by the Coxeter system

$$\hat{E}^{(i)}R_{j+1}R_{2n-j-1}, \hat{E}^{(i)}R_{j+2}R_{2n-j-2}, \dots, \hat{E}^{(i)}R_1R_{2n-1}, \text{ and } \hat{E}^{(i)}R_0$$

of type C_{n-j} . These generators coincide with the ϕ -images of the simple reflections in the statement of the lemma. \square

The case $i = 0$ of the above lemma confirms that the restriction of ϕ to $W(C_n)$ is an embedding of this group into $W(A_{2n-1})$ whose image coincides with the σ -fixed elements of $W(A_{2n-1})$.

The $W(A_{2n-1})$ -orbit of $B_{i,p}$ contains B_i , but, for $p < i$, these two admissible sets are in distinct $W(C_n)$ -orbits: For $B \in \mathcal{B}$, the numbers i , the size of B , and p , the number of roots in B fixed by σ , are constant on the $W(C_n)$ -orbit of B in \mathcal{A}_σ . They actually determine this orbit uniquely.

Proposition 5.15. *Let $B \in \mathcal{A}_\sigma$ be such that the number of σ -fixed roots in B is equal to p and such that B has cardinality i . Then there exists an element w of the subgroup $W(C_n)$ of $W(A_{2n-1})$ such that $wB_{i,p} = B$.*

Proof. By Theorem 3.9 and the case $i = 0$ of Lemma 5.14, it suffices to find a symmetric diagram w in $\text{BrM}(A_{2n-1})$ without horizontal strands, moving $B_{i,p}$ to B .

For each $\gamma \in B$ with $\gamma \neq \sigma(\gamma)$, where $\gamma = \alpha_t + \alpha_{t+1} + \dots + \alpha_s$, $1 \leq t \leq s \leq 2n-1$, we draw four vertical strands in w as follows: from $(t, 1)$ to $(k, 0)$, from $(2n+1-t, 1)$ to $(2n+1-k, 0)$, from $(s+1, 1)$ to $(k+1, 0)$, and from $(2n-s, 1)$ to $(2n-k, 0)$ with $\{\alpha_k, \alpha_{2n-k}\} \subset B_{i,p}$. For each $\gamma \in B$ with $\gamma = \sigma(\gamma)$, where $\gamma = \alpha_{n-t} + \alpha_{n-t+1} + \dots + \alpha_{n+t}$, $0 \leq t \leq n-1$, we draw two vertical strands: from $(n-t, 1)$ to $(n-k, 0)$, and from $(n+t+1, 1)$ to $(n+k+1, 0)$ where $\alpha_{n-k} + \alpha_{n-k+1} + \dots + \alpha_{n+k} \in B_{i,p}$. Between the remaining $2n-2i$ dots at the top and $2n-2i$ nodes at the bottom, we just draw vertical strands in such a way that these strands do not cross. This provides the required diagram w . \square

Proposition 5.16. *The homomorphism $\phi : \text{Br}(C_n) \rightarrow \text{SBr}(A_{2n-1})$ is surjective.*

Proof. It suffices to prove the statement for the corresponding monoids. So, let a be an element of $\text{BrM}(A_{2n-1})$ with $\sigma(a) = a$. Then, by Theorem 3.9, up to replacing a by a power of δ , we have

$$a = U\hat{E}^{(i)}VW \quad (5.8)$$

for some $i \in \{0, \dots, n\}$, $U \in \text{BrM}(A_{2n-1})\hat{E}^{(i)}$, $W \in \hat{E}^{(i)}\text{BrM}(A_{2n-1})$, and $V \in K_i$ such that $\text{ht}(U) + \text{ht}(V) + \text{ht}(W) = \text{ht}(a)$. Here B_i and K_i are as in Definition 5.13. As noted before, $\sigma(B_i) = B_i$, which implies $\sigma(\hat{E}^{(i)}) = \hat{E}^{(i)}$. From Lemma 5.12, it follows that $\hat{E}^{(i)} \in \phi(\text{BrM}(C_n))$. Now ϕ is a homomorphism, so it suffices to show that U, V, W are in the image of ϕ .

As $\sigma(a) = a$ we find $U\hat{E}^{(i)}VW = \sigma(U)\hat{E}^{(i)}\sigma(V)\sigma(W)$ with $\sigma(U) \in \text{BrM}(A_{2n-1})\hat{E}^{(i)}$, $\sigma(W) \in \hat{E}^{(i)}\text{BrM}(A_{2n-1})$ and $\sigma(V) \in K_i$ (note that $\sigma(K_i) = K_i$ by Lemma 5.14) such that $\text{ht}(\sigma(U)) + \text{ht}(\sigma(V)) + \text{ht}(\sigma(W)) = \text{ht}(a)$. According to Theorem 3.9, the expression in (5.8) is unique, which implies $\sigma(U) = U$, $\sigma(V) = V$, and $\sigma(W) = W$. From Lemma 5.14, we find $V \in \phi(\text{BrM}(C_n))$. Writing $U\hat{E}^{(i)}VW = (U\hat{E}^{(i)})V(W\hat{E}^{(i)})^{\text{op}}$ and using Proposition 4.2 (observe that the anti-involution $x \mapsto x^{\text{op}}$ commutes with ϕ and σ), we may restrict ourselves to the case where $a = U$ with $\text{ht}(U) = \text{ht}(a\emptyset)$. Therefore, we will assume that a is of this kind.

Put $B = a\emptyset$. Then $B \in \mathcal{A}_\sigma$ and so there are $w \in W(A_{2n-1})_\sigma = \phi(W(C_n))$ and $p, i \in \{0, 1, \dots, n\}$ with $0 \leq p \leq i$ and $i - p$ even such that $B = wB_{i,p}$.

If $B = B_{i,p}$, then, according to Lemma 5.11 and Theorem 3.9, $U = \phi(e_{p+1}e_{p+3} \cdots e_{i-1})\hat{E}^{(i)}$, which belongs to $\phi(\text{BrM}(C_n))$. Denote this element by $E^{(i,p)}$.

Now, in the general case, $wE^{(i,p)} = U\hat{E}^{(i)}V$ for some $V \in K_i$ such that $\text{ht}(V) = \text{ht}(wE^{(i,p)}) - \text{ht}(B)$. By Theorem 3.9 the element V of K_i is uniquely determined by w , so $U\hat{E}^{(i)}V = wE^{(i,p)} = \sigma(w)\sigma(E^{(i,p)}) = \sigma(wE^{(i,p)}) = U\hat{E}^{(i)}\sigma(V)$ implies $V = \sigma(V)$. The inverse V' of V in the group K_i with unit $\hat{E}^{(i)}$ satisfies $wE^{(i,p)}V' = U\hat{E}^{(i)} = U = a$. As V' is again uniquely determined by V , we have $\sigma(V') = V'$ and so Lemma 5.14 gives $V' \in \phi(\text{BrM}(C_n))$. We conclude $a = wE^{(i,p)}V' \in \phi(\text{BrM}(C_n))$. \square

6 Admissible sets and their orbits

We continue with the study of the Brauer monoid $\text{BrM}(C_n)$ of type C_n acting on \mathcal{A}_σ , the subset of \mathcal{A} of σ -invariant admissible sets. This leads to a normal form for elements of $\text{BrM}(C_n)$ to the extent that we can provide an upper bound on the rank of $\text{Br}(C_n)$. The bound found in Theorem 6.9 is instrumental in the proof at the end of this section of the main Theorem 1.1.

Lemma 6.1. *Let $i \in \{0, \dots, n\}$ and $p \in \{0, \dots, i\}$ be such that $q = (i - p)/2$ is an integer. Then the $W(C_n)$ -orbit of $B_{i,p}$ has size $n!/(p!q!(n - i)!)$.*

Proof. By Proposition 5.15, the cardinality of the orbit of $B_{i,p}$ under $W(C_n)$ is equal to the number of diagram tops with i horizontal strands of which precisely p strands are fixed by σ . This number is readily seen to be

$$\binom{n}{p} \binom{n-p}{2q} (4q-2)(4q-6) \cdots 2 = \frac{n!}{p!q!(n-i)!}.$$

□

Corollary 6.2. *The rank a_{2n} of $\text{SBr}(A_{2n-1})$ satisfies*

$$a_{2n} = \sum_{i=0}^n \left(\sum_{p+2q=i} \frac{n!}{p!q!(n-i)!} \right)^2 2^{n-i} (n-i)!.$$

Proof. If in a symmetric diagram with $2i$ horizontal strands, all horizontal strands are fixed, the remaining $2(n-i)$ vertical strands will be in one to one correspondence with the elements of the Weyl group of type C_{n-i} and order $2^{n-i}(n-i)!$. Therefore the corollary follows from Lemma 6.1. □

Now we proceed to describe the stabilizer in $W(C_n)$ of $B_{i,p}$.

Definition 6.3. Let $i \in \{0, \dots, n\}$ and $p \in \{0, \dots, i\}$ be such that $i-p = 2q$ for some $q \in \mathbb{N}$. By $A_{i,p}$ we denote the subgroup of $W(C_n)$ generated by the following elements:

$$\begin{aligned} r_j \quad (j = 0, \dots, p-1), \\ r_{p+2k-1} \quad (k = 1, \dots, q), \\ t_{0,p} = y_{p+1} r_{p+1} y_{p+1}, \\ t_{k,p} = r_{p+2k} r_{p+2k-1} r_{p+2k+1} r_{p+2k} \quad (k = 1, \dots, q-1). \end{aligned}$$

Furthermore, by L_i we denote the subgroup of $W(C_n)$ generated by $y_{i+1}, r_{i+1}, \dots, r_{n-1}$. Finally, we set $N_{i,p} = \langle A_{i,p}, L_i \rangle$ and let $D_{i,p}$ be a fixed set of representatives for left cosets of $N_{i,p}$ in $W(C_n)$.

Figure 4 depicts $B_{6,2}$ as a top and two elements of the form $t_{k,p}$.

It is easy to check that the generators of $A_{i,p}$ and L_i , and hence the whole group $N_{i,p}$ leaves $B_{i,p}$ invariant. The next lemma shows that $N_{i,p}$ is the full stabilizer of $B_{i,p}$ in $W(C_n)$.

Lemma 6.4. *The subgroup of $W(C_n)$ generated by $\{t_{j,p}\}_{j=0}^{q-1}$ is isomorphic to $W(C_q)$ and the cardinality of $A_{i,p}$ is $2^i p! q!$. Moreover, L_i is isomorphic to $W(C_{n-i})$. Furthermore, $N_{i,p}$ is the stabilizer of $B_{i,p}$ in $W(C_n)$ and isomorphic to $A_{i,p} \times L_i$.*

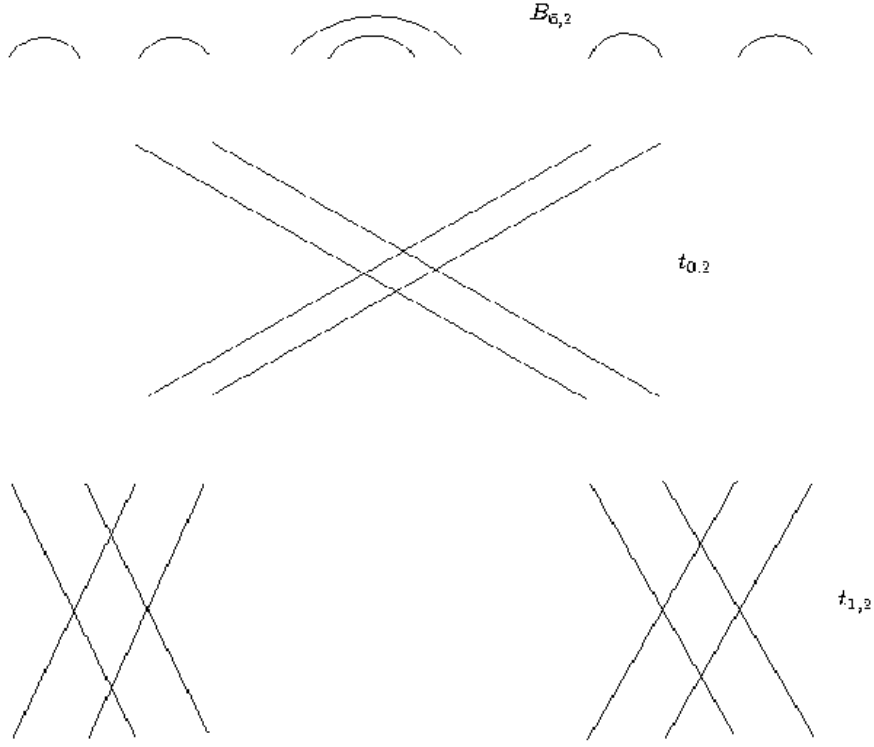


Figure 4: The ϕ -images of the elements $t_{0,2}$ and $t_{1,2}$.

Proof. Put

$$\begin{aligned} A &= \langle r_0, r_1, \dots, r_{p-1} \rangle, \\ B &= \langle t_{0,p}, t_{1,p}, \dots, t_{q-1,p} \rangle, \\ C &= \langle r_{p+1}, r_{p+3}, \dots, r_{i-1} \rangle. \end{aligned}$$

Being a parabolic subgroup of type C_p , the group A is isomorphic to $W(C_p)$. Since the supports of the simple reflections involved in A lie in $\{0, \dots, p-1\}$ and those of $B \cup C$ lie in $\{p+1, \dots, i\}$, each element of A commutes with each

element of $B \cup C$. Now we claim that B is isomorphic to $W(C_q)$. Ignoring the $2p$ vertical strands in the middle of generators of B in $\text{BrM}(A_{2n-1})$, and comparing them with canonical generators of $W(C_q)$ in $\text{BrM}(A_{2q-1})$ gives an easy pictorial proof of our claim.

Consider the diagrams of elements of B and C in $\text{BrM}(A_{2n-1})$. Each diagram of B only has non-crossing strands starting from $(n-k-1, 1)$ and $(n-k+1, 1)$, for $k = p+1, p+3, \dots, i-1$, which can never occur in nontrivial elements of C . Hence $B \cap C = \{1\}$. For the generators of B and C , the following equations hold.

$$\begin{aligned} t_{0,p} r_{p+2k-1} t_{0,p} &= r_{p+2k-1}, \text{ for } 1 \leq k \leq q \\ t_{k,p} r_{p+2k-1} t_{k,p} &= r_{p+2k+1}, \text{ for } 1 \leq k \leq q \\ t_{s,p} r_{p+2k-1} t_{s,p} &= r_{p+2k-1}, \text{ for } 1 \leq s, k \leq q, \text{ and } |s-k| > 1. \end{aligned}$$

Therefore the subgroup BC in $W(C_n)$ is the semiproduct of C and B with C normal. Consider the diagrams of elements of BC and A in $\text{Br}(A_{2n-1})$. Each element in BC keeps the $2p$ strands in the middle invariant, but each element of A keeps the left $2n-2p+2$ strands invariant. Therefore $A \cap BC = \{1\}$. Thus, $A_{i,p} = BC \times A$, and hence

$$|A_{i,p}| = |B||C||A| = 2^i p! q!.$$

The reflections $y_{i+1}, r_i, r_{i+1}, \dots, r_{n-1}$ have roots $\beta_0 + 2\beta_1 + \dots + 2\beta_i, \beta_{i+1}, \dots, \beta_{n-1}$, respectively, which form a simple root system of type C_{n-i} . Therefore the subgroup L_i is isomorphic to $W(C_{n-i})$.

In the Coxeter diagram A_{2n-1} we see that $A_{i,p} \cap L_i = 1$ and all elements in L_i commute with all elements in $A_{i,p}$, so $N_{i,p}$ is the direct product of L_i and $A_{i,p}$. This gives

$$|D_{i,p}| = \frac{|W(C_n)|}{|A_{i,p}||L_i|} = \frac{n!}{p!q!(n-i)!}.$$

By Lagrange's Theorem, the cardinality of $D_{i,p}$ is equal to the size of the $W(C_n)$ -orbit of $B_{i,p}$. Therefore by Lemma 6.1, $N_{i,p}$ is the stabilizer of $B_{i,p}$ in $W(C_n)$. \square

The study of the stabilizer of $B_{i,p}$ will now be used to rewrite products of $b_{p,i,p'}$ with elements of $W(C_n)$. The result is in Lemma 6.6 and needs the following special cases.

Lemma 6.5. *Let $i \in \{0, \dots, n\}$ and $p \in \{0, \dots, i\}$ with $i-p$ even.*

(i) *For each $r \in A_{i,p}$ we have $rb_{p,i,i} = b_{p,i,i}$.*

(ii) *For each $v \in L_i$ we have $ve_{i,p} = e_{i,p}v$ and $vb_{p,i,i} = b_{p,i,i}v$.*

Proof. (ii). By Lemma 4.3 and Definition 2.1, the two equations hold for the generators of L_i . Therefore they hold for each element of L_i .

(i). The roots β_j and $r_j r_{j-1} \cdots r_1 \beta_0$ are as in Proposition 5.9(iii), so

$$r_j z_j z_{j+1} = (r_j z_j r_j z_j) r_j \stackrel{(2.13)}{=} z_j (e_j z_j r_j) \stackrel{(2.18)}{=} z_j e_j z_j \stackrel{(4.5)}{=} z_j r_j z_j r_j = z_j z_{j+1}.$$

This proves that (i) is satisfied with $r = r_j$ for $j = 0, \dots, p-1$. For the choices $r = r_{p+2k-1}$ for $k = 1, \dots, q$ this is straightforward. Moreover, $t_{0,p} e_{p+1} = y_{p+1} (r_{p+1} y_{p+1} e_{p+1}) = y_{p+1} y_{p+1} e_{p+1} = e_{p+1}$, and

$$\begin{aligned} t_{k,p} e_{p+2k-1} e_{p+2k+1} &= r_{p+2k} r_{p+2k+1} r_{p+2k-1} r_{p+2k} e_{p+2k-1} e_{p+2k+1} \\ &\stackrel{(2.9)}{=} (r_{p+2k} r_{p+2k+1} e_{p+2k}) e_{p+2k-1} e_{p+2k+1} \\ &\stackrel{(2.9)+(2.7)}{=} (e_{p+2k+1} e_{p+2k} e_{p+2k+1}) e_{p+2k-1} \\ &\stackrel{(2.32)}{=} e_{p+2k-1} e_{p+2k+1}. \end{aligned}$$

So (i) holds for all generators of $A_{i,p}$ and hence for all of $A_{i,p}$. \square

Lemma 6.6. *Suppose $r \in W(C_n)$. Let $i \in \{0, \dots, n\}$ and $p \in \{0, \dots, i\}$ with $i - p$ even.*

(i) *There are $u \in D_{i,p}$ and $v \in L_i$ such that $rb_{p,i,i} = ub_{p,i,i}v$.*

(ii) *There are $u' \in D_{i,p}^{\text{op}}$ and $v' \in L_i$ such that $b_{i,i,p}r = v'b_{i,i,p}u'$.*

Proof. Let $r \in W(C_n)$. By Lemma 6.4 and Definition 6.3 for $D_{i,p}$, there exist $u \in D_{i,p}$, $v \in L_i$, and $a \in A_{i,p}$ such that $r = uva$. By Lemma 6.5,

$$rb_{p,i,i} = uvab_{p,i,i} = uvb_{p,i,i} = ub_{p,i,i}v.$$

The second statement follows by applying Proposition 4.2 to (i). \square

Our next step towards a normal form for elements of $\text{BrM}(C_n)$ is to describe products of elements from $W(C_n)b_{p,i,p'}W(C_n)$ with generators e_j . To this end we first prove two useful equalities.

Lemma 6.7. *In $\text{Br}(C_n)$, the following hold for $i \in \{2, \dots, n-1\}$.*

$$e_i z_{i+1} = e_i z_i, \tag{6.1}$$

$$e_{i-1} z_{i+1} z_i z_{i-1} = r_i r_{i-1} e_i z_i z_{i+1} z_{i-1}, \tag{6.2}$$

$$e_i z_{i+1} z_i e_i = \delta^2 e_i. \tag{6.3}$$

Proof. By Lemma 5.9(iii), $e_i z_{i+1} \stackrel{(4.7)}{=} e_i r_i z_i r_i \stackrel{(2.2)}{=} e_i z_i r_i \stackrel{(2.18)}{=} e_i z_i$. This proves (6.1).

Now (6.3) follows from Lemma 5.9(i) as

$$e_i z_{i+1} z_i e_i \stackrel{(6.1)}{=} e_i z_i^2 e_i = \delta e_i z_i e_i \stackrel{(2.16)}{=} \delta^2 e_i.$$

As for (6.2), note that (2.13) and Proposition 5.9(iii) give $z_{i-1} e_{i-1} z_{i-1} = r_{i-1} z_{i-1} r_{i-1} z_{i-1} = z_i z_{i-1}$. Hence

$$\begin{aligned} r_{i-1} r_i e_{i-1} z_{i+1} z_i z_{i-1} &\stackrel{(2.9)}{=} e_i e_{i-1} z_{i+1} z_i z_{i-1} \stackrel{4.3}{=} e_i z_{i+1} e_{i-1} z_i z_{i-1} \stackrel{(6.1)}{=} \delta e_i z_i e_{i-1} z_{i-1} \\ &\stackrel{(6.1)+4.2}{=} \delta e_i z_{i-1} e_{i-1} z_{i-1} = \delta e_i z_i z_{i-1} \stackrel{(6.1)}{=} e_i z_{i+1} z_i z_{i-1}, \end{aligned}$$

and the equation follows by left multiplication with $r_i r_{i-1}$. \square

The detailed information stated in the last sentence of the following proposition will be needed for the proof of cellularity of $\text{Br}(C_n, R, \delta)$ in the next section.

Proposition 6.8. *Let i, p, p' be natural numbers with $0 \leq p, p' \leq i \leq n$ and $i - p$ and $i - p'$ even. For each root $\beta \in \Psi^+$, there are $h \in \{i, i+1, i+2\}$, $k, m, m' \in \mathbb{N}$, $u \in D_{h,m}^{\text{op}}$, $w \in D_{h,m'}^{\text{op}}$, and $v \in L_h$ such that*

$$e_\beta b_{p,i,p'} = \delta^k u b_{m,h,m'} v w.$$

Moreover, if $h = i$, then $w = 1$ and $m' = p'$, while k, u , and v do not depend on p' .

Proof. We first sketch the general idea of proof. There are only two possible root lengths in the Coxeter root system Ψ of type C_n . We call $\beta \in \Psi$ short if $(\beta, \beta) = 1$ and long otherwise, in which case $(\beta, \beta) = 2$. In each case, only one root needs to be considered, for all other roots of the same length are conjugate to this particular one under the natural action of $W(C_n)$, and Lemma 6.6 can be applied to reduce to the representative root.

The top of $b_{p,i,p'}$ is the admissible set $B_{i,p}$ displayed in Figure 3.

First suppose β is long. Then β can be written as $\beta_0 + 2\beta_1 + \cdots + 2\beta_{t-1}$, for some $1 \leq t \leq n$, and so $e_\beta = z_t$. We will distinguish cases according to relations among t, i , and p , and apply induction on t and i . In Figure 5 the roots of the Cases L1, L2, and L3 are displayed as the top, middle, and bottom horizontal strand, respectively.

Case L1. Suppose $t \geq i + 1$. For any $t > i + 1$, we have $z_t = s z_{i+1} s^{-1}$, where $s = r_{t-1} \cdots r_{i+1}$. This implies that $z_t b_{p,i,p'} = s z_{i+1} b_{p,i,p'} s^{-1}$, with $s \in L_i$. Lemmas 6.6 and 6.5 can be used to reduce this case to the case where $t = i + 1$.

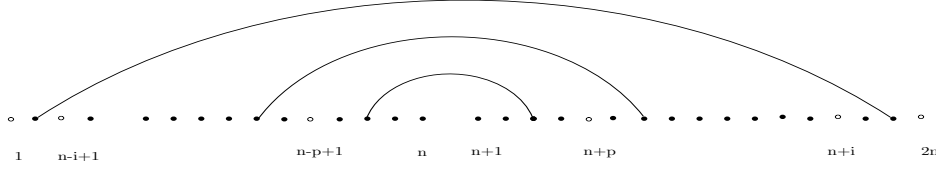


Figure 5: Horizontal strands representing the three cases for a long root β .

If $p = p' = i$, then $e_{i,p} = e_{i,p'} = 1$ and $z_{i+1}b_i = b_{i+1}$ by the definition of b_i in (5.1), as required.

If $p \neq i = p'$, it suffices to prove the equation

$$z_{i+1}b_{p,i,i} = rb_{p+1,i+1,i+1},$$

with r in the subgroup of $W(C_n)$ generated by r_0, r_1, \dots, r_i . We proceed by induction on i . In view of Lemma 6.7, (below IH is short for Inductive Hypothesis),

$$\begin{aligned} z_{i+1}b_{p,i,i} &\stackrel{(5.1)+(5.4)}{=} z_{i+1}e_{i-1}e_{i-2,p}z_i z_{i-1}b_{i-2} \stackrel{4.3}{=} e_{i-1}z_{i+1}z_i z_{i-1}e_{i-2,p}b_{i-2} \\ &\stackrel{(6.2)}{=} r_i r_{i-1}e_i z_i z_{i+1}z_{i-1}e_{i-2,p}b_{i-2} \stackrel{4.3}{=} r_i r_{i-1}e_i(z_{i-1}b_{p,i-2,i-2})z_i z_{i+1} \\ &\stackrel{\text{IH}}{=} r_i r_{i-1}e_i g b_{p+1,i-1,i-1}z_i z_{i+1} = r_i r_{i-1}g e_i b_{p+1,i-1,i-1}z_i z_{i+1} \\ &= r_i r_{i-1}g b_{p+1,i+1,i+1}, \end{aligned}$$

where g is an element of the subgroup of $W(C_n)$ generated by r_0, r_1, \dots, r_{i-2} . Hence the claim holds.

The case $p = i \neq p'$ now follows by use of Proposition 4.2.

If $p, p' \neq i$ then, by the above,

$$\begin{aligned} z_{i+1}b_{p,i,p'} &= r e_{i+1,p+1}b_{i+1}e_{i,p'} = r e_{i+1,p+1}b_i e_{i,p'} z_{i+1} = r e_{i+1,p+1}b_{i+1}e_{i+1,p'+1}r' \\ &= r b_{p+1,i+1,p'+1}r', \end{aligned}$$

where $r, r' \in W(C_n)$. By Lemma 6.5, we conclude that this expression can be written in the required form with $h = i + 1$.

Case L2. Next suppose $p + 1 \leq t < i + 1$. By definition of z_i ,

$$\begin{aligned} z_{i-1} &= r_{i-1}z_i r_{i-1}, \\ z_{i-2} &= r_{i-2}r_{i-3}r_{i-1}r_{i-2}z_i r_{i-2}r_{i-3}r_{i-1}r_{i-2}, \end{aligned}$$

with $r_{i-1}, r_{i-2}r_{i-3}r_{i-1}r_{i-2} \in A_{i,p}$. By induction on t , we can find $r \in A_{i,p}$ such that $z_t = r z_i r^{-1}$. By Lemma 6.5,

$$e_\beta b_{p,i,p'} = z_t b_{p,i,p'} = r z_i r^{-1} b_{p,i,p'} = r z_i b_{p,i,p'}.$$

In view of Lemma 6.6, this reduces the problem to rewriting $z_i b_{p,i,p'}$ in the required form.

Now

$$\begin{aligned} z_i e_{i-1} z_i z_{i-1} &= (r_{i-1} z_{i-1} r_{i-1} e_{i-1}) z_{i-1} z_i \stackrel{(2.2)+(4.1)}{=} z_{i-1} e_{i-1} z_{i-1} z_i \\ &\stackrel{(2.13)}{=} (r_{i-1} z_{i-1} r_{i-1}) z_{i-1} z_i = z_i z_{i-1} z_i \\ &= \delta z_i z_{i-1}, \end{aligned}$$

so

$$\begin{aligned} z_i b_{p,i,i} &\stackrel{(5.4)}{=} z_i e_{i-1} e_{i-2,p} z_{i-1} z_i b_{i-2} \stackrel{4.3}{=} z_i e_{i-1} z_{i-1} z_i e_{i-2,p} b_{i-2} \\ &= \delta z_{i-1} z_i b_{p,i-2,i-2}, \end{aligned}$$

by the claim of Case L1 and Lemma 6.6, the above can be written as $u b_{p+2,i,i} v$ with $u \in D_{i,p+2}$ and $v \in L_i$, and so the proposition holds in this case as $z_i b_{p,i,p'} = u b_{p+2,i,p'} v$.

Case L3. We remain with the case where $1 \leq t \leq p$. We have

$$z_t b_{p,i,p'} = e_{i,p} z_t b_i e_{i,p'} = \delta e_{i,p} b_i e_{i,p'} = \delta b_{p,i,p'},$$

and so the proposition holds with $h = i$.

We next consider the case where β is a short root, which means $\beta = \beta_s + \beta_{s+1} + \cdots + \beta_t$ with $0 < s \leq t \leq n-1$ or $\beta = \beta_0 + 2\beta_1 + \cdots + 2\beta_{s-1} + \beta_s + \beta_{s+1} + \cdots + \beta_t$ with $0 \leq s \leq t \leq n-1$. We will distinguish seven cases by values of s and t corresponding to the horizontal strands of Figure 6. The seven cases occur in the order from top to bottom.

Case S1. Suppose that β is a linear combination of β_j ($j = i+1, \dots, n-1$). Then, by Lemma 6.7,

$$\begin{aligned} e_{i+1} b_{p,i,p'} &= e_{i,p} e_{i+1} b_i e_{i,p'} \stackrel{(6.3)}{=} \delta^{-2} e_{i,p} (e_{i+1} z_{i+2} z_{i+1} e_{i+1}) b_i e_{i,p'} \\ &= \delta^{-2} e_{i,p} e_{i+1} z_{i+2} z_{i+1} b_i e_{i+1} e_{i,p'} \\ &\stackrel{(5.1)+(5.4)}{=} \delta^{-2} e_{i+2,p} b_{i+2} e_{i+2,p'} = \delta^{-2} b_{p,i+2,p'}. \end{aligned}$$

At the same time, for any such $\beta \in \Psi^+$, there exists an element $r \in L_i$ such that $\beta = r \beta_{i+1}$, thus $e_\beta = r e_{i+1} r^{-1}$. Now $e_\beta b_{p,i,p'} = \delta^{-2} r b_{p,i+2,p'} r^{-1}$, and hence the proposition holds with $h = i+2$.

Case S2. Suppose $\beta = \beta_s + \beta_{s+1} + \cdots + \beta_t$, with $p \leq s \leq i \leq t \leq n-1$.

If $p = i$, then $e_{i,p} = 1$. First, consider the case $t = s = i$. Since

$$e_i z_{i+1} z_i \stackrel{(6.1)}{=} e_i z_i z_i \stackrel{5.9(i)}{=} \delta e_i z_i,$$

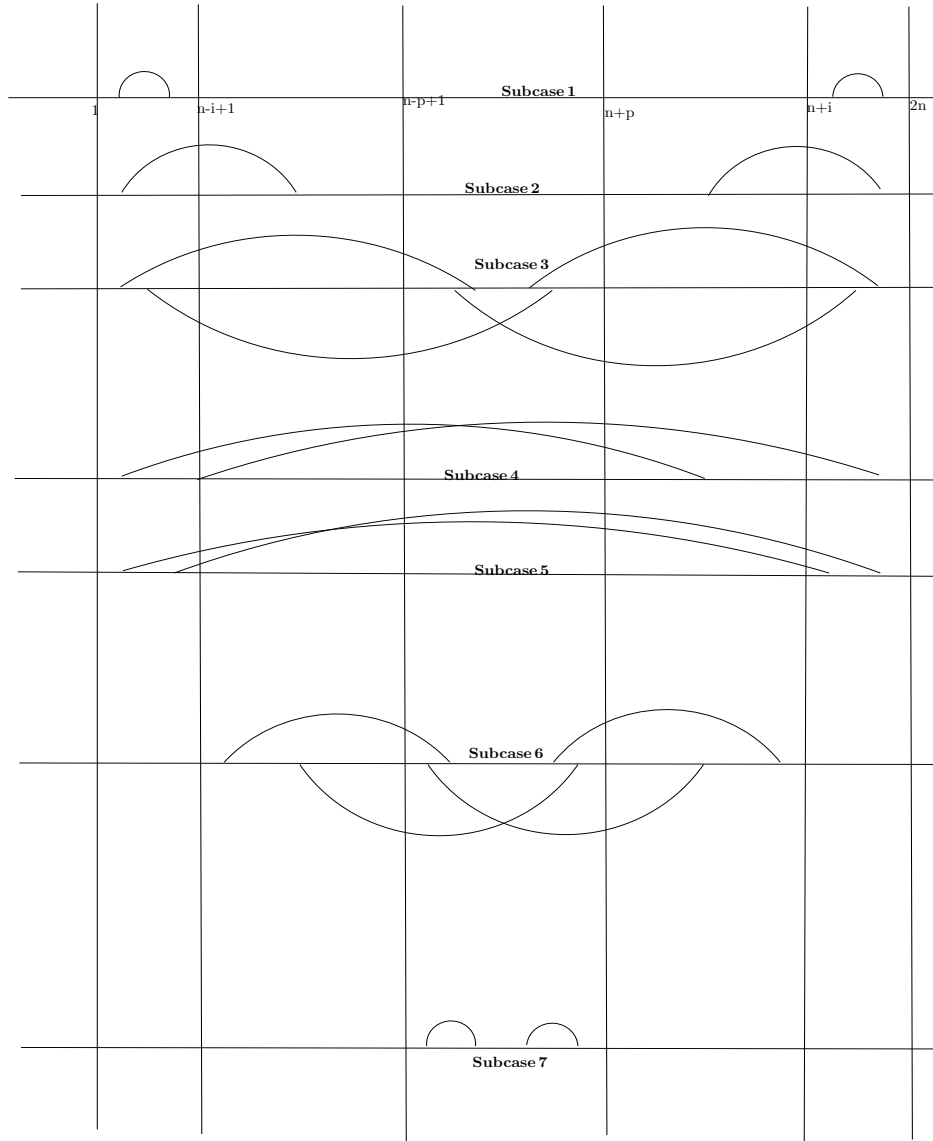


Figure 6: Strands corresponding to 7 possibilities for the long root β .

the use of the claim of Case L1 and Proposition 4.2 gives the existence of an element $r \in W(C_n)$ such that

$$\begin{aligned} e_i b_{i,i,p'} &= \delta^{-1} e_i z_i b_i e_{i,p'} \stackrel{5.9(i)}{=} \delta^{-1} e_i z_{i+1} b_i e_{i,p'} \stackrel{4.3}{=} \delta^{-1} e_i b_{i,i,p'} z_{i+1} \\ &\stackrel{L1}{=} \delta^{-1} e_i b_{i+1,i+1,p'} r = \delta^{-1} b_{i-1,i+1,p'} r, \end{aligned}$$

as required.

If $t \neq s$, then $e_\beta = r_{\beta''} r_{\beta'} e_i r_{\beta'} r_{\beta''}$, where $\beta' = \beta_s + \dots + \beta_{i-1}$, $\beta'' = \beta_{i+1} + \dots + \beta_t$, which implies that $r_{\beta'} \in A_{i,p}$, $r_{\beta''} \in L_i$, and hence $e_\beta b_i e_{i,p'} = r_{\beta''} r_{\beta'} e_i b_i e_{i,p'} r_{\beta''}$. As in the argument for $e_i b_{i,i,p'}$ above, this can be written in the required form with $h = i + 1$.

On the other hand if $p \neq i$, then $e_{i,p} \neq 1$. Therefore for $\beta = \beta_s + \beta_{s+1} + \dots + \beta_t$, with $p \leq s \leq i \leq t \leq n - 1$, since there is some $l \in \{p+1, p+3, \dots, i-1\}$, such that $e_\beta e_l = r_l r_\beta e_l$, implying that $e_\beta e_{i,p} = r_l r_\beta e_{i,p}$ and $e_\beta b_{p,i,p'} = r_l r_\beta b_{p,i,p'}$. By Lemma 6.6, the proposition holds with $h = i$.

Case S3. Suppose $\beta = \beta_s + \beta_{s+1} + \dots + \beta_t$ or $\beta = \beta_0 + 2\beta_1 + \dots + 2\beta_{s-1} + \beta_s + \dots + \beta_t$ with $0 < s \leq p$ and $i \leq t \leq n - 1$.

First consider $\beta = \beta_p + \dots + \beta_i$. Following the argument for $e_i z_{i+1} z_i = \delta e_i z_i$ in the above case, we find that $e_\beta z_{i+1} z_i = \delta e_\beta z_i$ holds, which implies

$$e_\beta b_{p,i,p'} = e_{i,p} e_\beta b_i e_{i,p'} = \delta^{-1} e_{i,p} e_\beta b_{i+1} e_{i,p'}. \quad (6.4)$$

Observe that

$$\begin{aligned} r_{i-1} r_i e_\beta e_{i-1} &= e_{\beta-\beta_i-\beta_{i-1}} r_{i-1} r_i e_{i-1} r_i r_{i-1} r_i \stackrel{(2.10)}{=} e_{\beta-\beta_i-\beta_{i-1}} e_i r_{i-1} r_i, \\ \text{and } r_{i-1} r_i b_{i+1} &\stackrel{6.5}{=} b_{i+1}. \end{aligned}$$

Therefore (6.4) can be written as

$$\begin{aligned} \delta^{-1} e_{i,p} e_\beta b_{i+1} e_{i,p'} &= \delta^{-1} (e_{i-1} r_i r_{i-1}) e_{i-2,p} e_{\beta-\beta_i-\beta_{i-1}} b_{i+1} e_{i,p'} \\ &\stackrel{(2.10)}{=} \delta^{-1} r_i r_{i-1} e_i e_{i-2,p} e_{\beta-\beta_i-\beta_{i-1}} b_{i+1} e_{i,p'} \\ &= \delta^{-1} r_i r_{i-1} e_i (e_{\beta-\beta_i-\beta_{i-1}} e_{i-2,p} b_{i-2}) z_{i-1} z_i z_{i+1} e_{i,p'}. \end{aligned}$$

By induction on i , we can use an argument as in the claim of Case S1, and the above can be written as

$$\begin{aligned} \delta^{-1} r_i r_{i-1} e_i g e_{i-1,p-1} b_{i-1} z_{i-1} z_i z_{i+1} e_{i,p'} &= r_i r_{i-1} g e_i e_{i-1,p-1} b_{i-1} z_i z_{i+1} e_{i,p'} \\ &= r_i r_{i-1} g e_{i+1,p-1} b_i z_{i+1} e_{i,p'} \\ &= r_i r_{i-1} g e_{i+1,p-1} b_{i,i,p'} z_{i+1}, \end{aligned}$$

where $g \in W(C_n)$ is a product of elements from r_0, r_1, \dots, r_{i-2} . By Case L1 and Proposition 4.2, the proposition holds with $h = i + 1$.

We return to the general setting of Case S3. Then there exists $r' \in L_i$ and $r'' \in A_{i,p}$ such that $r'r''\hat{\beta} = \beta$, with $\hat{\beta} = \beta_p + \cdots + \beta_i$. Then

$$e_\beta b_{p,i,p'} = r'r''e_{\hat{\beta}} b_{p,i,p'} r',$$

hence the proposition holds in Case S3 due to Lemma 6.6.

Case S4. If $\beta = \beta_0 + 2\beta_1 + \cdots + 2\beta_{s-1} + \beta_s + \cdots + \beta_t$ with $p \leq s \leq i \leq t \leq n-1$, and let $\beta' = \beta_s + \cdots + \beta_t$. Then $e_\beta = y_s e_{\beta'} y_s$. When $p = i = s$, we see that $y_s \in A_{i,p}$; when $p \neq i$, there is some $l \in \{p+1, p+3, \dots, i-1\}$, such that $e_\beta e_l = r_l r_\beta e_l$. This brings us back to the argument of Case S2.

Case S5. If $\beta = \beta_0 + 2\beta_1 + \cdots + 2\beta_s + \beta_{s+1} + \cdots + \beta_t$, with $i < s \leq t \leq n-1$, then $\beta = y_{s+1} \beta'$, with $\beta' = \beta_{s+1} + \beta_{s+2} + \cdots + \beta_t$, and $y_{s+1} \in L_i$. Therefore

$$e_\beta b_{p,i,p'} = y_{s+1} (e_{\beta'} b_{p,i,p'}) y_{s+1},$$

and we are back in Case S1.

Case S6. If $\beta = \beta_0 + 2\beta_1 + \cdots + 2\beta_s + \beta_{s+1} + \cdots + \beta_t$ or $\beta = \beta_s + \beta_{s+1} + \cdots + \beta_t$, with $0 \leq s \leq t$ and $p \leq t \leq i-1$, then there must be some $e_j \in \{e_{p+2j-1}\}_{j=1}^{(i-p)/2}$ such that β is not orthogonal to β_j , or $\{\beta, \beta_j\}$ is not admissible, or $\beta = \beta_j$. Then, by Lemma 5.9, there exists $r \in W(C_n)$ such that $e_\beta e_{i,p} = r e_{i,p}$ or $e_\beta e_{i,p} = \delta r e_{i,p}$. This implies that the proposition holds with $h = i$.

Case S7. If β can be written as a linear combination of $\{\beta_j\}_{j=0}^{p-1}$, then β is conjugate to β_{p-1} under the subgroup of $A_{i,p}$ generated by $\{r_j\}_{j=0}^{p-1}$. Then we can find a $r \in A_{i,p}$ such that $r\beta_{p-1} = \beta$, which implies

$$e_\beta b_{p,i,p'} = r e_{p-1} r^{-1} b_{i,i,p'} \stackrel{6.5(i)}{=} r e_{p-1} b_{i,i,p'} = r b_{p-2,i,p'},$$

so the proposition holds with $h = i$ due to Lemma 6.6. \square

Theorem 6.9. *Each element in the monoid $\text{BrM}(C_n)$ can be written as*

$$\delta^k u b_{p,i,p'} v w^{\text{op}},$$

where $k \in \mathbb{Z}$ and $i, p, p' \in \{0, \dots, n\}$ with $i - p$ and $i - p'$ even, $u \in D_{i,p}$, $v \in L_i$, and $w \in D_{i,p'}$. In particular, $\text{Br}(C_n)$ is free of rank at most a_{2n} .

Proof. Let U be the set of elements of $\text{BrM}(C_n)$ of the indicated form. We show that U is invariant under left multiplication by generators of $\text{BrM}(C_n)$. To this end, consider an arbitrary element $a = \delta^k u b_{p,i,p'} v w^{\text{op}}$ of U . Obviously $\delta^{\pm 1} a \in U$. Without loss of generality, we may take $k = 0$.

Let $r \in W(C_n)$. By Lemma 6.6 applied to ru there are $u' \in D_{i,p}$ and $v' \in L_i$ such that $ra = r u b_{p,i,p'} v w^{\text{op}}$. By Lemma 6.5, this is equal to $u' b_{p,i,p'} v' v w^{\text{op}}$

and, as $v'v \in L_i$, the set U is invariant under left multiplication by Weyl group elements.

Finally, consider the generator e_j of $\text{BrM}(C_n)$. Writing $\beta = u^{\text{op}}\alpha_j$ we have $e_j a = e_j u b_{p,i,p'} v w^{\text{op}} = u e_\beta b_{p,i,p'} v w^{\text{op}}$ and by Proposition 6.8 this belongs to U again.

Now, by Proposition 4.2 we also find that U is invariant under right multiplication by generators. This proves that U is invariant under both left and right multiplication by any generator of $\text{BrM}(C_n)$. As it contains the identity $(b_{0,0,0})$, it follows that U coincides with the whole monoid.

As for the last assertion of the theorem, observe that freeness of $\text{Br}(C_n)$ over $\mathbb{Z}[\delta^{\pm 1}]$ is immediate from the fact that it is a monomial algebra with a finite number of generators. By the first assertion, its rank is at most

$$\sum_{i=0}^n \left(\sum_{p \equiv i \pmod{2}} |D_{i,p}| \right)^2 \cdot |L_i|.$$

By Lemma 6.1, the cardinality of $D_{i,p}$ is $n!/(p!q!(n-i)!)$, where $q = (i-p)/2$, and, by Lemma 6.4, L_i is isomorphic to $W(C_{n-i})$, which has $2^{n-i}(n-i)!$ elements. Therefore, the rank of $\text{Br}(C_n)$ over $\mathbb{Z}[\delta^{\pm 1}]$ is at most

$$\sum_i \left(\sum_{p,q:p+2q=i} \frac{n!}{p!q!(n-i)!} \right)^2 2^{n-i}(n-i)!.$$

Corollary 6.2 gives that this sum is equal to a_{2n} . □

We are now ready to prove Theorem 1.1. Theorem 6.9 shows that $\text{Br}(C_n)$ is free of rank at most a_{2n} . By Proposition 5.16, the homomorphism $\phi : \text{Br}(C_n) \rightarrow \text{SBr}(A_{2n-1})$ is surjective, so the rank of $\text{Br}(C_n)$ is at least the rank of $\text{SBr}(A_{2n-1})$, which is known to be a_{2n} by Corollary 2.7. Thus, the ranks of $\text{Br}(C_n)$ and $\text{SBr}(A_{2n-1})$ coincide and ϕ is an isomorphism.

7 Further properties of type C algebras

In this section we prove that the algebra $\text{BrM}(C_n, R, \delta)$ is cellular, in the sense of Graham and Lehrer [10], provided R is an integral domain containing the inverse to 2. The proof given here runs parallel to the proof of the corresponding result for D_n in [5, Section 6]. We finish by discussing a few more desirable properties of the newly found Brauer algebras.

Recall from [10] that an associative algebra A over a commutative ring R is cellular if there is a quadruple $(\Lambda, T, C, *)$ satisfying the following three conditions.

(C1) Λ is a finite partially ordered set. Associated to each $\lambda \in \Lambda$, there is a finite set $T(\lambda)$. Also, C is an injective map

$$\prod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \rightarrow A$$

whose image is an R -basis of A .

(C2) The map $*$: $A \rightarrow A$ is an R -linear anti-involution such that $C(x, y)^* = C(y, x)$ whenever $x, y \in T(\lambda)$ for some $\lambda \in \Lambda$.

(C3) If $\lambda \in \Lambda$ and $x, y \in T(\lambda)$, then, for any element $a \in A$,

$$aC(x, y) \equiv \sum_{u \in T(\lambda)} r_a(u, x)C(u, y) \pmod{A_{<\lambda}},$$

where $r_a(u, x) \in R$ is independent of y and where $A_{<\lambda}$ is the R -submodule of A spanned by $\{C(x', y') \mid x', y' \in T(\mu) \text{ for } \mu < \lambda\}$.

Such a quadruple $(\Lambda, T, C, *)$ is called a *cell datum* for A . We will describe such a quadruple for $\text{Br}(C_n, R, \delta)$.

For $*$ we will use the anti-involution op determined in Proposition 4.2. Let $i \in \{0, \dots, n\}$. By Theorem 6.9, each element in the monoid $\text{BrM}(C_n)$ can be written in the form

$$\delta^k u b_{p,i,p'} v w^{\text{op}},$$

where $k \in \mathbb{Z}$ and $i, p, p' \in \{0, \dots, n\}$ are such that $i - p$ and $i - p'$ are even, $u \in D_{i,p}$, $v \in L_i$, and $w \in D_{i,p'}$. As the coefficient ring R is an integral domain containing the inverse of 2, it satisfies the conditions of [8, Theorem 1.1], so by [8, Corollary 3.2] the group rings $R[L_i]$ ($i = 0, \dots, n$) are all cellular. By Lemma 6.4, the subalgebra RL_i of $\text{Br}(C_n, R, \delta)$ (with unit $\hat{b}^{(i)}$, see (5.6)) generated by L_i is isomorphic to $R[L_i]$. Let $(\Lambda_i, T_i, C_i, *_i)$ be a cell datum for RL_i as in [8]. Observe that the generators of L_i in Definition 6.3 are fixed by op , so RL_i is op -invariant. By [8, Section 3], $*_i$ is the map op on RL_i and so $*_i$ is the restriction of op to RL_i .

The underlying poset Λ will be $\{B_i\}_{i=0}^n$, as defined in (5.2). We say that $B_i > B_j$ if and only if $i < j$ or, equivalently, $B_i \subset B_j$. In particular, \emptyset is the greatest element of Λ .

The set $T(B_i)$ is taken to be the set of all triples $(u, e_{i,p}, s)$ where $u \in D_{i,p}$ (see Definition 6.3), $p \in \{0, \dots, i\}$ with $i - p$ even, and the product $e_{i,p}$ is given by (5.4), and $s \in T_i$. Clearly, this set is finite.

The map C is given by $C((u, e_{i,p}, s), (w, e_{i,p'}, t)) = u e_{i,p} C_i(s, t) e_{i,p'} w^{\text{op}}$. By Lemma 6.5, $u b_{p,i,p'} v w^{\text{op}} = (u e_{i,p})(b_i v)(w e_{i,p'})^{\text{op}}$, so the image of C is a basis by Theorems 1.1 and 6.9, and the fact that $\{C_i(s, t) \mid s, t \in T_i\}$ is a basis

for RL_i (which is a consequence of (C1) for $(\Lambda_i, T_i, C_i, *_i)$). This gives a quadruple $(\Lambda, T, C, *)$ satisfying (C1).

For (C2) notice that $(ue_{i,p}C_i(s, t)e_{i,p'}w^{\text{op}})^{\text{op}} = we_{i,p'}C_i(s, t)^{\text{op}}e_{i,p}u^{\text{op}}$. Now $C_i(s, t)^{\text{op}} = C_i(t, s)$, by the cellularity condition (C2) for RL_i and so (C2) holds for the cell datum $(\Lambda, T, C, *)$.

Finally, we check condition (C3) for $(\Lambda, T, C, *)$. It suffices to consider the left multiplications by r_j and e_j of $ue_{i,p}C_i(s, t)e_{i,p'}w^{\text{op}}$. Up to linear combinations, we can replace the latter expression by $ue_{i,p}b_i v e_{i,p'}w^{\text{op}}$ for $v \in L_i$. Now, by Lemma 6.6 for r_j and Proposition 6.8 for e_j (note the product lies in $\text{Br}(C_n, R, \delta)_{<B_i}$ if $h > i$), (C3) holds for the cell datum $(\Lambda, T, C, *)$. Therefore we have now proved

Theorem 7.1. *Let R be an integral domain with $2^{-1} \in R$. Then the quadruple $(\Lambda, T, C, *)$ is a cell datum for $\text{Br}(C_n, R, \delta)$, proving the algebra is cellular.*

We continue by discussing some desirable properties of the Brauer algebra $\text{Br}(C_n)$. First of all, for any disjoint union M of diagrams of type Q , for Q a simply laced graph, and C_k for $k \in \mathbb{N}$, the Brauer algebra is defined as the direct product of the Brauer algebras whose types are the components of X . The next result states that parabolic subalgebras behave well.

Proposition 7.2. *Let J be a set of nodes of the Dynkin diagram C_n . Then the parabolic subalgebra of the Brauer algebra $\text{Br}(C_n)$, that is, the subalgebra generated by $\{r_j, e_j\}_{j \in J}$, is isomorphic to the Brauer algebra of type J .*

Proof. In view of induction on $n - |J|$ and restriction to connected components of J , it suffices to prove the result for $J = \{1, \dots, n-1\}$ and for $J = \{0, \dots, n-2\}$. In the former case, the type is A_{n-1} and the statement follows from the observation that the symmetric diagrams without strands crossing the vertical line through the middle of the segments connecting the dots $(n, 1)$ and $(n+1, 1)$ are equal in number to the Brauer diagrams on the $2n$ nodes (realized to the left of the vertical line). In the latter case, the type is C_{n-1} and the statement follows from the observation that the symmetric diagrams with vertical strands from $(1, 1)$ to $(1, 0)$ and from $(2n, 1)$ to $(2n, 0)$ are equal in number to the symmetric diagrams related to $\text{BrM}(C_{n-1})$. \square

The reader may have wondered why the study of symmetric diagrams was restricted to type A_m for m odd. The answer is that, if $m = 2n$ is even, each symmetric diagram has a fixed vertical strand from the dot $(n, 1)$ to $(n, 0)$. The removal of this strand leads to an isomorphism of the algebra of the symmetric diagrams with $\text{SBr}(A_{2n-1})$, and so this construction provides no new algebra. This is remarkable in that the root system obtained by projecting Φ onto the σ -fixed subspace of the reflection representation, as in Definition 5.1, leads to a root system of type B_n instead of C_n .

On the other hand, the presentation by generators and relations given in Definition 2.1 suggests, at least when $\delta = 1$, the definition of the Brauer algebra of type B_n as for C_n , but with the roles of 0 and 1 reversed in defining relations (2.11)–(2.18). The dimensions for the Brauer algebras $\text{Br}(B_n, R, 1)$ with $n \leq 5$ thus obtained were found to be

n	1	2	3	4	5
a_n	3	25	273	3801	66315

It is likely that these algebras emerge with a construction similar to the one given for C_n but with A_{2n-1} replaced by D_{n+1} and σ by the diagram automorphism interchanging the two short end nodes. This will be the subject of further investigation.

At the time of writing of this paper, Chen [3] presented a definition of a generalized Brauer algebra of type $I_2(m)$. For $m = 4$, this type coincides with C_2 , but Chen’s algebra has dimension $2m + m^2 = 24$, whereas our $\text{Br}(C_2)$ has dimension 25.

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References

- [1] S. Bigelow, Braid groups are linear, *Journal of the American Mathematical Society*, **14** (2001), 471–486.
- [2] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Annals of Mathematics*, **38** (1937), 857–872.
- [3] Z. Chen, Algebras associated with pseudo reflection groups: A generalization of Brauer algebras, [arXiv:1003.5280v1](https://arxiv.org/abs/1003.5280v1), March 2010.
- [4] A.M. Cohen, B. Frenk and D.B. Wales, Brauer algebras of simply laced type, *Israel Journal of Mathematics*, **173** (2009) 335–365.
- [5] A.M. Cohen, D.A.H. Gijsbers and D.B. Wales, The BMW Algebras of type D_n , [arXiv:0704.2743](https://arxiv.org/abs/0704.2743), April 2007.

- [6] J. Crisp, Injective maps between Artin groups, in Geometric Group Theory Down Under, Lamberra 1996 (J. Cossey, C.F. Miller III, W.D. Neumann and M.Shapiro, eds.) De Gruyter, Berlin, 1999, 119–137.
- [7] T. tom Dieck, Quantum groups and knot algebra, Lecture notes, May 4, 2004.
- [8] M. Geck, Hecke algebras of finite type are cellular, *Inventiones Mathematicae*, **169** (2007), 501–517.
- [9] J. J. Graham, Modular representations of Hecke algebras and related algebras, Ph. D. thesis, University of Sydney, 1995.
- [10] J.J. Graham and G.I. Lehrer, Cellular algebras, *Inventiones Mathematicae* **123** (1996), 1–44.
- [11] B. Mühlherr, Coxeter groups in Coxeter groups, pp. 277–287 in *Finite Geometry and Combinatorics* (Deinze 1992). London Math. Soc. Lecture Note Series **191**, Cambridge University Press, Cambridge, 1993.